TMA4130 Matematikk 4N Problem Sheet: 5th Week

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We shall be using the following definitions of the Fourier transform and convolution, respectively:

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} \, \mathrm{d}x, \qquad f * g(x) := \int_{\mathbb{R}} f(x-y) g(y) \, \mathrm{d}y.$$

1. Compute the following integrals (Problems 1 and 3 in cap. 11 sec. 7):

(i) (there is a typo in the book here)

$$\frac{1}{\pi} \int_0^\infty \frac{\cos(xw) + w \sin(xw)}{1+w^2} \, \mathrm{d}w$$

(ii)

$$\frac{2}{\pi} \int_0^\infty \frac{1-\cos(\pi w)}{w} \sin(xw) \, \mathrm{d}w$$

2. Find the Fourier transform to the followng (cap. 11 sec. 9 problems 6, 9, and cap. 11 sec. 10):

(i) $f(x) = e^{-\lambda x^2}$, where $\lambda > 0$, by completing-the-square in the exponent, and assuming $\int_{\mathbb{R}} e^{-x^2/a^2} \, \mathrm{d}x$ is some consant C_a,

(ii) $f(x) = \mathbb{1}_{[0,2\pi]}(x)$,

(iii) $f(x) = e^{-\lambda|x|}$, where $\lambda > 0$,

(iv) $f(x) = |x| \mathbb{1}_{[-1,1]}(x)$.

Recall that $\mathbb{1}_E$ is the characteristic function defined by $\mathbb{1}_E(x) = 1$ if $x \in E$ and nought otherwise.

3.

(i) Using formula (4) in cap. 11 sec. 9, for $f : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} |f|^2 \, \mathrm{d}x < \infty$, show that the double transform is

$$\hat{\hat{f}}(x) = f(-x).$$

(ii) With reference to 2(iii) above, or otherwise, find the Fourier transform of

$$f(\xi) = \frac{1}{\xi^2 + \lambda^2}.$$

(iii) (partial exam question in 2016 for 4M) Compute the convolution $(f * f)(x)$.

4†. Define the two sided-Laplace transform as the Laplace transform where the integral is taken over $\mathbb{R}$:

$$(\mathcal{L}f)(s) = \int_{\mathbb{R}} f(t) e^{-st} \, \mathrm{d}t$$

To get the usual Laplace transform just put $f(t)u(t)$ in place of $f(t)$, where $u(t)$ is the Heaviside function.

Using formula (4) in cap. 11 sec. 9, show that disregarding conditions for exchanging limits, the Mellin Inversion Formula for the (two sided-)Laplace transform holds. In particular, show that if $\mathcal{L}f = F$, F is smooth, and the following limits hold

$$\frac{1}{2\pi i} \lim_{T\to\infty} \int_\gamma^{\gamma \pm iT} F(s) e^{st} \, \mathrm{d}s = \frac{1}{2\pi i} \int_\gamma^{\gamma \pm i\infty} F(s) e^{st} \, \mathrm{d}s,$$

where $\gamma \in \mathbb{R}$ is fixed (to be greater than the real part of all singularities of F if it isn't smooth), then

$$f(t) = (\mathcal{L}^{-1}F)(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} \, \mathrm{d}s.$$

(hint: the integral on the right can be re-written using a simple substitution $s \mapsto \gamma + i\tau$ *where* γ *is still a fixed constant so that:*

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,\mathrm{d}s = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\gamma+i\tau)e^{\gamma t}e^{i\tau t}\,\mathrm{d}\tau,$$

and all that needs to be done is to put in the definition of $F = \mathcal{L}f$ *as an integral.)*