TMA4130 Matematikk 4N Solutions: 4th Week

1.

(i) neither, even, odd , even , even
(ii) $e^x = \cosh(x) + \sinh(x)$ — $\cosh(x)$ is even, $\sinh(x)$ is odd.
(iii) For each non-zero ε, the value is 0. Hence the limit is 0.

2.

(i) We consider the nth mode of the equation

$$y_n'' + cy_n' + y_n = a_n \cos(nt) + b_n \sin(nt).$$

We can write y_n as $y_n = A_n \cos(nt) + B_n \sin(nt)$. Differentiating we get

$$\begin{aligned} cy_n' =& cB_n n \cos(nt) - cA_n n \sin(nt) \\ y'' =& - A_n n^2 \cos(nt) - B_n n^2 \sin(nt), \end{aligned}$$

and hence

$$\begin{aligned} A_n(1-n^2) + cB_n n =& a_n \\ B_n(1-n^2) - cA_n n =& b_n. \end{aligned}$$

This gives

$$A_n = \frac{a_n(1-n^2) - b_n cn}{(1-n^2)^2 - (cn)^2}, \qquad B_n = \frac{a_n cn + b_n(1-n^2)}{(cn)^2 + (1-n^2)^2}.$$

(ii) There is no dissipation mechanism, but continual energy input – therefore there is no steady state in the limit only blowup. In partiular, because the resonance frequency is 1, energy concentrates in the first mode and $|A_1|, |B_1| \to \infty$.

3. Recall that in the last sheet, assuming the convergence result, using $f(x) = \sum_{n\in\mathbb{Z}} g(x - 2n\pi)$ for $g(x) = |x|^2 \mathbb{1}_{[-\pi,\pi)}(x)$, we arrived at

$$x^2 = \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(nx) + \frac{1}{3}\pi^2.$$

on $[-\pi, \pi)$.

Applying Parseval's identity from the book directly,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \, \mathrm{d}x = 2a_0^2 + \sum_{n\geq 1} a_n^2 + b_n^2 = \frac{2}{9}\pi^4 + \sum_{n\geq 1} \frac{16}{n^4}.$$

Manipulating the above to leave only the infinite series on one side and a number on the other gives the correct result.

4. Apply the Parseval's identity to the function $(f+g)$ and use the linearity of the Fourier transform: Since we are dealing with real-valued functions,

$$|f+g|^2 = |f|^2 + 2fg + |g|^2.$$

The Fourier transform $\mathscr{F} f$ and $\mathscr{F} g$ are *complex-valued*, however. And for two complex numbers z and w,

$$|z+w|^2 = (z+w)(\bar{z} + \bar{w}) = z\bar{z} + w\bar{w} + z\bar{w} + w\bar{z} = |z|^2 + |w|^2 + 2\mathfrak{Re}(z\bar{w}).$$

Here $\bar{z}$ denotes the complex conjugate of z.

Using Parseval's identity on $f+g$, we have

$$\begin{aligned}
&\sum_n |(\mathscr{F}f)(n)|^2 + |(\mathscr{F}g)(n)|^2 + 2\mathfrak{Re}\{(\mathscr{F}f)(n)\overline{(\mathscr{F}g)(n)}\} \\
=&\sum_n |\mathscr{F}f(n) + \mathscr{F}g(n)|^2 \\
=&\sum_n |\mathscr{F}(f+g)|^2 \\
=&C\int_0^{2\pi} |f+g|^2(x)\ \mathrm{d}x \\
=&C\int_0^{2\pi} |f|^2(x) + |g|^2(x) + 2f(x)g(x)\ \mathrm{d}x.
\end{aligned}$$

Now apply Parseval's identity to f and g separately to see that

$$\sum_n \mathfrak{Re}\{(\mathscr{F}f)(n)\overline{(\mathscr{F}g)(n)}\} = C\int_0^{2\pi} f(x)g(x)\ \mathrm{d}x.$$

Now notice that from the definition of the Fourier transform, since f and g are real-valued functions, $\overline{(\mathscr{F}g)(n)} = (\mathscr{F}g)(-n)$.

This way of deriving off-diagonal results from the diagonal case is usually known as the POLARIZATION TRICK, which is generally applicable to bilinear/multilinear structures.