TMA4130 Matematikk 4N Problem Sheet: 4th Week

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1.

(i) Problems 1 and 2 of cap. 11 sec. 2 :
Are the following functions even or odd, or neither?

$$e^x, \quad e^{-|x|}, \quad x^2\tan(\pi x), \quad \sin(x^2), \quad x\cot(x).$$

(ii) For functions neither even nor odd in (i), decompose them into a sum of one even and one odd function.

(iii) Evaluate the limit:

$$\lim_{\varepsilon \searrow 0} \int_{[-M,M]\setminus(-\varepsilon,\varepsilon)} \frac{1}{x}\,\mathrm{d}x, \qquad M > 0.$$

The arrow "$\searrow$" indicates a limit from above, and $[-M,M]\setminus(-\varepsilon,\varepsilon)$ is the interval $[-M,M]$ *without* the interval $(-\varepsilon,\varepsilon)$.

2.

(i) Problem 13 of cap. 11 sec. 3.:
Find the steady-state oscillations of $y'' + cy' + y = r(t)$ with $c > 0$ and

$$r(t) = \sum_{n=1}^{N} a_n\cos(nt) + b_n\sin(nt).$$

(ii) Problem 3 of cap. 11 sec. 3:
What happens in (i) as $c \to 0$ if a_n and b_n up to $n = N$ are all non-zero?

3. Problem 12 of cap. 11 sec. 4. : With reference to Problem 4 on Problem Sheet 3, or Problem 11 of cap. 11 sec. 2, use Parseval's identity on $f(x) = \sum_{n\in\mathbb{Z}}(x-2n\pi)^2\mathbb{1}_{[-\pi,\pi)}(x-2n\pi)$ to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

4. Define the Fourier coefficients of a function f to be

$$(\mathscr{F}f)(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,\mathrm{d}x.$$

(This is c_n in the book, but we indicate explicit dependence on the function.) In this definition of the Fourier transform, Parseval's identity reads:

$$\int_0^{2\pi} |f(x)|^2\,\mathrm{d}x = C\sum_{n\in\mathbb{Z}} |(\mathscr{F}f)(n)|^2,$$

where C is some absolute positive constant (i.e., independent of f).

A 2π-periodic function $f : \mathbb{T} \to \mathbb{R}$ is SQUARE SUMMABLE if

$$\int_0^{2\pi} |f(x)|^2\,\mathrm{d}x < \infty.$$

Use Parseval's identity to show that for some (possibly different) absolute positive constant C, and two real-valued, 2π-periodic, square summable functions f and g,

$$\sum_n \mathfrak{Re}\{(\mathscr{F}f)(n)(\mathscr{F}g)(-n)\} = C\int_0^{2\pi} f(x)g(x)\,\mathrm{d}x,$$

where $\mathfrak{Re}\{z\}$ denotes the real part of z.

(Hint: consider the function $h(x) = f(x) + g(x)$.)