TMA4130 Matematikk 4N Solutions: 3rd Week

1.

(i) The boundedness conditions simply ensures that the convolution exist. Taking the Laplace transform of the equation we arrive at

$$(s^2+2s-8)Ły - sa - a = \mathcal{L}r.$$

Re-arranging we have:

$$\mathcal{L}y = \frac{1}{(s-2)(s+4)}\mathcal{L}r + a\frac{(s+1)}{(s-2)(s+4)}.$$

Using partial fractions we find:

$$\frac{1}{(s-2)(s+4)} = \frac{1/6}{(s-2)} - \frac{1/6}{(s+4)}$$
$$\frac{s+1}{(s-2)(s+4)} = \frac{1}{s+4} - \frac{3}{(s-2)(s+4)} = \frac{-1/2}{(s-2)} + \frac{3/2}{(s+4)}.$$

The inverse Laplace transforms are:

$$\mathcal{L}^{-1}\left\{\frac{1}{(\cdot-2)(\cdot+4)}\right\} = \frac{1}{6}\big(e^{2t} - e^{-4t}\big)$$
$$\mathcal{L}^{-1}\left\{\frac{(\cdot+1)}{(\cdot-2)(\cdot+4)}\right\} = \frac{1}{2}\big(-e^{2t} + 3e^{-4t}\big)$$

Therefore,

$$y(t) = \frac{1}{6}\int_0^t r(t-v)\big(e^{2v} - e^{-4v}\big)\,\mathrm{d}v + \frac{a}{2}\big(-e^{2t} + 3e^{-4t}\big).$$

(ii) Taking a derivative we see that

$$\log\left(\frac{s}{s-1}\right) = -\int_s^\infty \frac{1}{r(r-1)}\,\mathrm{d}r.$$

Using partial fractions we see that

$$\frac{-1}{s(s-1)} = \frac{1}{s} - \frac{1}{s-1}.$$

The inverse Laplace transform of the above is

$$f(t) = 1 - e^t.$$

Therefore the inverse Laplace transform of $\log(s/(s-1))$ is

$$(e^t - 1)/t.$$

2.

(i) Applying the Laplace transform,

$$s\begin{pmatrix}\mathcal{L}y_1\\ \mathcal{L}y_2\end{pmatrix} = \begin{pmatrix}1 & 1\\ -5 & -3\end{pmatrix}\begin{pmatrix}\mathcal{L}y_1\\ \mathcal{L}y_2\end{pmatrix} + \begin{pmatrix}y_1(0)\\ y_2(0)\end{pmatrix}.$$

Re-arranging we get:

$$\begin{aligned}\begin{pmatrix}\mathcal{L}y_1\\ \mathcal{L}y_2\end{pmatrix} &= \begin{pmatrix}s-1 & -1\\ 5 & s+3\end{pmatrix}^{-1}\begin{pmatrix}y_1(0)\\ y_2(0)\end{pmatrix}\\ &=\frac{1}{s^2+2s+2}\begin{pmatrix}s+3 & 1\\ -5 & s-1\end{pmatrix}\begin{pmatrix}y_1(0)\\ y_2(0)\end{pmatrix}\\ &=\frac{1}{(s+1)^2+1}\begin{pmatrix}s+3 & 1\\ -5 & s-1\end{pmatrix}\begin{pmatrix}y_1(0)\\ y_2(0)\end{pmatrix}.\end{aligned}$$

Using partial fractions, the inverse transforms are:

$$\begin{aligned}\frac{1}{(s+1)^2+1} &\overset{\mathcal{L}^{-1}}{\rightsquigarrow} e^{-t}\sin(t) =: A\\ \frac{s-1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{2}{(s+1)^2+1} &\overset{\mathcal{L}^{-1}}{\rightsquigarrow} e^{-t}\cos(t) - 2e^{-t}\sin(t) =: B\\ \frac{s+3}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{2}{(s+1)^2+1} &\overset{\mathcal{L}^{-1}}{\rightsquigarrow} e^{-t}\cos(t) + 2e^{-t}\sin(t) =: C.\end{aligned}$$

The solution is therefore:

$$\begin{aligned}y_1(t) &= y_1(0)C + y_2(0)A\\ y_2(t) &= -5y_1(0)A + y_2(0)B.\end{aligned}$$

Subsituting in the functions we arrive at:

$$\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix} = \begin{pmatrix}y_1(0)\\ y_2(0)\end{pmatrix}e^{-t}\cos(t) + \begin{pmatrix}2y_1(0)+y_2(0)\\ -5y_1(0)-2y_2(0)\end{pmatrix}e^{-t}\sin(t).$$

(ii) If $(y_1(0), y_2(0)) \neq 0$, as $t \to \infty$, because the real parts of the exponents in both eigenfunctions are negative, $(y_1(0), y_2(0)) \to (0,0)$ nevertheless.

(iii) Again we first apply the Laplace transform directly:

$$s\begin{pmatrix}\mathcal{L}y_1\\ \mathcal{L}y_2\end{pmatrix} = \begin{pmatrix}4 & -2\\ 3 & -1\end{pmatrix}\begin{pmatrix}\mathcal{L}y_1\\ \mathcal{L}y_2\end{pmatrix} + \begin{pmatrix}y_1(0)\\ y_2(0)\end{pmatrix}.$$

Next we re-arrange the equation:

$$\begin{aligned}\begin{pmatrix}\mathcal{L}y_1\\ \mathcal{L}y_2\end{pmatrix} &= \begin{pmatrix}s-4 & 2\\ -3 & s+1\end{pmatrix}^{-1}\begin{pmatrix}y_1(0)\\ y_2(0)\end{pmatrix}\\ &=\frac{1}{(s-1)(s-2)}\begin{pmatrix}s+1 & -2\\ 3 & s-4\end{pmatrix}\begin{pmatrix}y_1(0)\\ y_2(0)\end{pmatrix}\end{aligned}$$

Using partial fractions, the inverse transforms are:

$$\begin{aligned}\frac{1}{(s-1)(s-2)} &= \frac{1}{s-2} - \frac{1}{s-1} \overset{\mathcal{L}^{-1}}{\rightsquigarrow} -e^{t} + e^{2t} =: A\\ \frac{s+1}{(s-1)(s-2)} &= \frac{-2}{s-1} + \frac{3}{s-2} \overset{\mathcal{L}^{-1}}{\rightsquigarrow} -2e^{t} + 3e^{2t} =: B\\ \frac{s-4}{(s-1)(s-2)} &= \frac{3}{s-1} - \frac{2}{s-2} \overset{\mathcal{L}^{-1}}{\rightsquigarrow} 3e^{t} - 2e^{2t} =: C.\end{aligned}$$

The solution is therefore:

$$\begin{aligned}y_1(t) &= y_1(0)B - 2y_2(0)A\\ y_2(t) &= 3y_1(0)A + y_2(0)C.\end{aligned}$$

Subsituting in the functions we arrive at:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} -2y_1(0) + 2y_2(0) \\ -3y_1(0) + 3y_2(0) \end{pmatrix} e^t + \begin{pmatrix} 3y_1(0) - 2y_2(0) \\ 3y_1(0) - 2y_2(0) \end{pmatrix} e^{2t}.$$

(iv) Notice that by settng either vector to $\binom{0}{0}$, $(y_1(0), y_2(0)) = (0,0)$ is the only solution of the simultaneous equations. As both exponentials tend to infinity as $t \to \infty$, if $(y_1(0), y_2(0)) \neq (0,0)$, then the solution tends to infinity as $t \to \infty$.

3.

(i) This is example 11.2.5 in the book, with a shift.

Obviously if $f(x) = h(x + \alpha)$, and f is 2π-periodic, then

$$\int_{-\pi}^{\pi} f(x)e^{-inx}\, \mathrm{d}x = e^{in\alpha} \int_{-\pi}^{\pi} h(x)e^{-inx}\, \mathrm{d}x.$$

Write h for the function in example 11.2.5. Our function f is given by $h(x - \pi) = 2\pi f(x)$. In the book they derived

$$h(x) = \pi - 2\sum_{n\geq 1} \frac{\sin(nx)}{n}(-1)^n = \pi - 2\sum_{n\geq 1} \frac{e^{inx} - e^{-inx}}{2in}(-1)^n.$$

So

$$f(x) = \frac{1}{2}e^{i0\cdot\pi} - \frac{1}{\pi}\sum_{n\geq 1} \frac{e^{inx}e^{in\pi} - e^{-inx}e^{-in\pi}}{2in}(-1)^n = \frac{1}{2} - \frac{1}{\pi}\sum_{n\geq 1}\frac{\sin(nx)}{n}.$$

(ii) This is an even function on $[-\pi, \pi)$, therefore it suffices to compute the cosine coefficients (the remaining are nought):

$$\begin{aligned}
\pi a_n &= \int_{-\pi}^{\pi} |t| \cos(nt)\, \mathrm{d}t \\
&= -\int_{-\pi}^{0} t\cos(nt)\, \mathrm{d}t + \int_{0}^{\pi} t\cos(nt)\, \mathrm{d}t \\
&= \frac{-1}{n} t\sin(nt)\Big|_{-\pi}^{0} + \int_{-\pi}^{0} \frac{1}{n}\sin(nt)\, \mathrm{d}t + \frac{1}{n} t\sin(nt)\Big|_{-\pi}^{0} - \int_{0}^{\pi} \frac{1}{n}\sin(nt)\, \mathrm{d}t \\
&= \frac{-1}{n^2}\cos(nt)\Big|_{-\pi}^{0} + \frac{1}{n^2}\cos(nt)\Big|_{0}^{\pi} \\
&= ((-1)^n - 1)\frac{2}{n^2}
\end{aligned}$$

For a_0 we can integrate directly/use area formula for triangles to get $a_0 = \pi/2$.

4.

(i) It is easier to work directly with e^{-inx} here. For $n \neq 0$,

$$\begin{aligned}
\int_{-\pi}^{\pi} x^2 e^{-inx}\, \mathrm{d}x &= \frac{-1}{in} x^2 e^{-inx}\Big|_{-\pi}^{\pi} + \frac{2}{in}\int_{-\pi}^{\pi} xe^{-inx}\, \mathrm{d}x \\
&= \frac{-1}{in} x^2 e^{-inx}\Big|_{-\pi}^{\pi} + \frac{2}{n^2} xe^{-inx}\Big|_{-\pi}^{\pi} - \frac{2}{n^2}\int_{-\pi}^{\pi} e^{-inx}\, \mathrm{d}x \\
&= 0 + \frac{4\pi}{n^2}(-1)^n - 0,
\end{aligned}$$

as $e^{in\pi} = e^{-in\pi} = (-1)^n$, and $\int_{-\pi}^{\pi} e^{-inx}\, \mathrm{d}x = 0$.

This is of course the cosine series which can be seen both from the fact that the function is even over $[-\pi, \pi)$ and the fact that result is real.

Therefore,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,\mathrm{d}x = \frac{4}{n^2}(-1)^n.$$

For $n = 0$, we have that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,\mathrm{d}x = \frac{1}{3}\pi^2.$$

(ii) We see that the function f satisfies the assumptions of the convergence theorem, therefore on $[-\pi, \pi)$,

$$x^2 = \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos(nx) + \frac{1}{3}\pi^2.$$

Putting $x = \pi$ gives us

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

If

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = c,$$

since

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n)^2} + \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2},$$

it holds that

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{3c}{4}.$$

Therefore,

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{3}{4}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{8}.$$