TMA4130 Matematikk 4N Problem Sheet: 3rd Week

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1.

(i) Let $r(t)$ be a bounded continuous function that is zero outside of $[-M, M]$, $M > 0$. Express the solution to the following problem as an integral:

$$y'' + 2y' - 8y = r, \qquad y(0) = -y'(0) = a.$$

(ii) Find the inverse Laplace transform of $F(s) = \log(s/(s-1))$.

2.

(i) Use the Laplace transform to find the general solution to the following system of ODEs:

$$\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= -5y_1 - 3y_2. \end{aligned}$$

(ii) With respect to (i), show that if $(y_1(0), y_2(0)) = (0,0)$, then $(y_1(t), y_2(t)) = (0,0)$. What happens to $(y_1(t), y_2(t))$ as $t \to \infty$ if $(y_1(0), y_2(0)) \neq (0,0)$?

(iii) Use the Laplace transform to find the general solution to the following system of ODEs:

$$\begin{aligned} y_1' &= 4y_1 - 2y_2 \\ y_2' &= 3y_1 - y_2. \end{aligned}$$

(iv) With respect to (iii), show that if $(y_1(0), y_2(0)) = (0,0)$, then $(y_1(t), y_2(t)) = (0,0)$. What happens to $(y_1(t), y_2(t))$ as $t \to \infty$ if $(y_1(0), y_2(0)) \neq (0,0)$?

3. Compute the Fourier coefficients associated with the following functions:

(i) $f(t) = \frac{t}{2\pi} - \left[\frac{t}{2\pi}\right]$, where, again, $[t]$ is the largest integer smaller than t,

(ii) $f(t) = \sum_{n\in\mathbb{Z}} g(t - 2n\pi)$, where $g(x) = |x|\mathbb{1}_{[-\pi,\pi)}(x)$. For a set $E \subseteq \mathbb{R}$, the function $\mathbb{1}_E : \mathbb{R} \to \{0,1\}$ is the characteristic function of E, defined to be

$$\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

This is sometimes alternatively denoted by $\chi_E(x)$ (for $\chi\alpha\rho\alpha\kappa\tau\acute{\eta}\rho$).

4.

(i) Compute the Fourier coefficients associated with $f(x) = \sum_{n\in\mathbb{Z}} g(t - 2n\pi)$, where $g(x) = x^2\mathbb{1}_{[-\pi,\pi)}(x)$.

(ii) Use the convergence theorem (Cap. 11.1, Theorem 2 *Representation by a Fourier Series*) and show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$