TMA4130 Matematikk 4N Solutions: 2nd Week

1.

(i) $f(t) = \sin(t)\cos(t) = \sin(2t)/2$. Therefore $(\mathcal{L}f)(s) = 1/(s^2+4)$.

(ii) We saw last time that for a periodic function of period T,

$$(\mathcal{L}f)(s) = \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}\,\mathrm{d}t.$$

Here $T = 1$, and on $t \in [0,1]$, $f(t) = t$, therefore,

$$(\mathcal{L}f)(s) = \frac{1}{1-e^{-s}}\left(\frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right).$$

(iii) Since $\mathcal{L}\{\cdot^{\alpha}\}(s) = \Gamma(\alpha+1)/s^{\alpha+1}$, by the first shifting theorem,

$$\mathcal{L}\{e^{\beta\cdot}\cdot^{\alpha}\}(s) = \Gamma(\alpha+1)/(s-\beta)^{\alpha+1},$$

and by the second shifting theorem, writing $f(t) = \exp(\beta^2)\exp(\beta(t-\beta))(t-\beta)^{\alpha}u(t-\beta)$,

$$(\mathcal{L}f)(s) = \Gamma(\alpha+1)e^{\beta^2}e^{-\beta s}(s-\beta)^{-\alpha-1}.$$

(iv) First write $\sin(r) = \sin(r-\beta+\beta) = \sin(r-\beta)\cos(\beta) + \cos(r-\beta)\sin(\beta)$.
Then by the convolution theorem,

$$(\mathcal{L}f)(s) = \mathcal{L}\{\cdot^{\alpha}\}(\cos(\beta)\cdot\mathcal{L}\{\sin(\cdot-\beta)u(\cdot-\beta)\} + \sin(\beta)\cdot\mathcal{L}\{\cos(\cdot-\beta)u(\cdot-\beta)\}).$$

Finally using the second shifting theorem, this yields

$$(\mathcal{L}f)(s) = \Gamma(\alpha+1)s^{-\alpha-1}e^{-\beta s}\cdot\left(\frac{\cos(\beta)}{s^2+1} + \frac{\sin(\beta)s}{s^2+1}\right).$$

2. This is not entirely a shifted data problem because one datum is initial and the other is terminal. These problems are not always determinate/solvable, but this one is. The point here is that you need to recognise what it is that you do not know — it is not y — you know how to find the general solution using Laplace's transform. What you really don't know is the value of $y'(0)$. Applying the Laplace transform, we have

$$(s^2+1)\mathcal{L}y - sy(0) - y'(0) = \int_0^1 te^{-st}\,\mathrm{d}t = \frac{-1}{s}te^{-st}\Big|_0^1 + \int_0^1 s^{-1}e^{-st}\,\mathrm{d}t = \frac{-1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}.$$

We know $y(0) = 0$. Therefore,

$$(\mathcal{L}y)(s) = \frac{\frac{-1}{s}e^{-s}(1+1/s) + 1/s^2 + y'(0)}{(s^2+1)}$$

$$y(t) = -(1-\cos(t-1))u(t-1) - ((t-1)-\sin(t-1))u(t-1) + (t-\sin(t)) + y'(0)\sin(t).$$

Take a (formal) derivative of y and evaluate it at $t = \pi$:

$$\begin{aligned} y'(\pi) &= -\sin(\pi-1)u(\pi-1) - (1-\cos(\pi-1))u(\pi-1) + (1-\cos(\pi)) + y'(0)\cos(\pi) \\ &= -\sin(\pi-1) + \cos(\pi-1) + 1 - y'(0). \end{aligned}$$

Set

$$K = \cos(\pi-1) - \sin(\pi-1)$$

We have $y'(0) = K$ and

$$y(t) = -(1-\cos(t-1))u(t-1) - ((t-1)-\sin(t-1))u(t-1) + (t-\sin(t)) + K\sin(t).$$

3.

(i) Both come from direct differentiation of an integral.

The point of this question is merely to give a hint for what to do for (ii). Let it be pointed out that (a) by the substitution $t - r = u$, either definition of convolution can be shown to be commutative and (b) the derivative and convolution by an integrable (of a differentiable function) commute with one another.

(ii) Since convolution and differentiation commute if the convolution is taken over all $\mathbb{R}$ (instead of over $[0, t]$), we can convolve each term of the first equation to get

$$\sum_{k=0}^{n} \frac{\mathrm{d}^k}{\mathrm{d}t^k}(x * f)(t) = (\delta * f)(t) = \int \delta(t-u) f(u)\, \mathrm{d}u = f(t).$$

Notice that since the initial data are all nought and $0 * f = 0$, we have that $y = (x * f)(t)$ solves the second equation.

4.

(i) Using the triangle inequality (and Fubini's theorem),

$$\begin{aligned}
\int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y) g(y)\, \mathrm{d}y \right| \mathrm{d}x &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y) g(y)|\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)|\, \mathrm{d}x\, |g(y)|\, \mathrm{d}y \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)|\, \mathrm{d}x\, |g(y)|\, \mathrm{d}y \\
&= \int |f(x)|\, \mathrm{d}x \cdot \int_{\mathbb{R}} |g(y)|\, \mathrm{d}y.
\end{aligned}$$

(ii) This is the same as (i):

$$\left| \int_{\mathbb{R}} f(x-y) g(y)\ dy \right| \leq \int_{R} |f(x-y)||g(y)|\, \mathrm{d}y \leq M \int_{\mathbb{R}} |f(x-y)|\, \mathrm{d}y = M \int_{\mathbb{R}} |f(x)|\, \mathrm{d}x.$$