TMA4130 Matematikk 4N Problem Sheet: 2nd Week

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1. Let u be the Heaviside/unit step function. Compute the Laplace transforms of the following functions:

(i) $f(t) = \sin(t)\cos(t)$,
(ii) $f(t) = t - [t]$, where $[t]$ is the *floor function*, i.e., $[t]$ is the largest integer smaller than t,
(iii) $f(t) = e^{\beta t}(t-\beta)^{\alpha} u(t-\beta)$ for $\alpha, \beta > 0$,
(iv) $f(t) = \int_0^t \sin(r) u(r-\beta)(t-r)^{\alpha}\, \mathrm{d}r$ for $\alpha, \beta > 0$.

2. Use the Laplace transform to solve the following mixed initial-terminal value data problem over $t \in [0, \pi]$:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y = \begin{cases} t & t \leq 1 \\ 0 & t > 1 \end{cases}, \qquad y(0) = 0, \qquad \frac{\mathrm{d}y}{\mathrm{d}t}(\pi) = 1.$$

3.

(i) Show that for f differentiable and bounded on $\mathbb{R}$ and g integrable (i.e., $\int_{\mathbb{R}} |g(x)|\, \mathrm{d}x < \infty$),

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} f(t-r) g(r)\, \mathrm{d}r = \int_{\mathbb{R}} f'(t-r) g(r)\, \mathrm{d}r = \int_{\mathbb{R}} f'(r) g(t-r)\, \mathrm{d}r,$$

and if additionally g is differentiable,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t f(t-r) g(r)\, \mathrm{d}r = f(0) g(t) + \int_0^t f'(t-r) g(r)\, \mathrm{d}r = g(t) f(0) + \int_0^t f(t-r) g'(r)\, \mathrm{d}r.$$

(ii) Let $a_0, \ldots, a_n$ be constants and let f be a smooth, integrable function. If it is known that the solution of

$$\sum_{k=0}^{n} a_k \frac{\mathrm{d}^k y}{\mathrm{d}t^k} = \delta(t), \qquad \forall k = 0, \ldots, n-1,\ y^{(k)}(0) = 0.$$

is given by $y = x(t)$, what is the solution of

$$\sum_{k=0}^{n} a_k \frac{\mathrm{d}^k y}{\mathrm{d}t^k} = f(t), \qquad \forall k = 0, \ldots, n-1,\ y^{(k)}(0) = 0?$$

This illustrates the importance of the impulse response of a system. We shall see this again when we study partial differential equations.

4.

(i) Prove that the family of integrable functions $\mathbb{R} \to \mathbb{R}$ form an algebra under the convolution product defined as:

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y)\, \mathrm{d}y.$$

In particular, show that if $\int_{\mathbb{R}} |f(x)|\, \mathrm{d}x$, $\int_{\mathbb{R}} |g(x)|\, \mathrm{d}x$ are bounded, then

$$\int_{\mathbb{R}} |(f * g)(x)|\, \mathrm{d}x \leq \int_{\mathbb{R}} |f(x)|\, \mathrm{d}x \cdot \int_{\mathbb{R}} |g(x)|\, \mathrm{d}x,$$

and hence $f * g$ remains an integrable function.

(ii) With reference to the convolution defined in (i), prove also that if $f(x)$ is integrable and $|g(x)|$ is uniformly bounded by $0 \leq M < \infty$, then the convolution $f * g$ is uniformly bounded over $\mathbb{R}$ thus:

$$|(f * g)(x)| \leq M \int_{\mathbb{R}} |f(x)|\, \mathrm{d}x.$$