TMA4130 Matematikk 4N Solutions: 1st Week

1.

(i) The Taylor series for $\cos(t)$ and $\sin(t)$ are absolutely convergent for fixed t, so they can be summed up term-by-term

$$\cos(t) + i\sin(t) = \sum_k \frac{(-1)^k t^{2k}}{(2k)!} + i \sum_n \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \sum_n \frac{(it)^n}{n!} = \exp(it).$$

Now polarise the identity: $e^{it} + e^{-it} = 2\cos(t)$ and $e^{it} - e^{-it} = 2i\sin(t)$.

(ii) $(e^{it} + e^{-it})^3 = e^{3it} + e^{-3it} + 3(e^{it} + e^{-it})$

(iii)

$$\mathcal{L}f = \mathcal{L}\{\cos(3\omega\cdot)\} + 3\mathcal{L}\{\cos(\omega\cdot)\} = \frac{s}{s^2+9\omega^2} + \frac{3s}{s^2+\omega^2}.$$

(iv) It tends to nought uniformly in $s > 0$ because for every ε, there is an $N(= 1/\varepsilon)$ such that if $\omega > N$,

$$\frac{\omega}{s^2+\omega^2} \leq \frac{1}{N} \leq \varepsilon.$$

2.

(i) This is the result of a direct calculation:

$$\begin{aligned}(\mathcal{L}f)(s) &= \int_0^\infty f(t)e^{-st}\,\mathrm{d}t \\ &= \sum_{N=0}^\infty \int_{NT}^{(N+1)T} f(t)e^{-st}\,\mathrm{d}t \\ &= \sum_{N=0}^\infty \int_0^T f(t+NT)e^{-s(t+NT)}\,\mathrm{d}t \\ &= \int_0^T f(t)e^{-st}\,\mathrm{d}t \cdot \sum_{N=0}^\infty e^{-sNT}.\end{aligned}$$

The sum is a geometric series and equals $1/(1-e^{-sT})$.

(ii) This is also a direct calculation:

$$\begin{aligned}(\mathcal{L}f)(s) &= \int_0^\infty f(t)e^{-st}\,\mathrm{d}t \\ &= \int_0^\infty \alpha^{-k-1} f(\alpha t)e^{-(s/\alpha)\cdot(\alpha t)}\,\mathrm{d}(\alpha t).\end{aligned}$$

Using the substitution $\alpha t \mapsto u$, we have

$$(\mathcal{L}f)(s) = \alpha^{-k-1}(\mathcal{L}f)(s/\alpha). \tag{1}$$

(iii) The point of this question is that $(\mathcal{L}f)(0) = \int f(t)\,\mathrm{d}t$. Taking $\alpha \to \infty$ we see that $\alpha^{-k-1} \to 0$. Therefore in order for the equality to hold where $(\mathcal{L}f)(s) \neq 0$, it must be that the other factor tends in magnitude to infinity. That is, $|(\mathcal{L}f)(0)| = \infty$ unless $f \equiv 0$.

(iv) Taking $\alpha \to 0$ from above in (1) above, we see that if $\mathcal{L}f$ is uniformly bounded, then $\lim_{s\to\infty}(\mathcal{L}f)(s) = 0$.

(v) This is theorem 3 in section 6.2 in the book:

$$\mathcal{L}g = \frac{1}{s}\mathcal{L}f.$$

3.

(i) Set $g(t) = \cos(t + 3\pi/2) = \sin(t)$. Since $f(t) = g(3t)$, we have

$$(\mathcal{L}f)(s) = \frac{1}{3}(\mathcal{L}g)(s/3) = \frac{1}{3}\frac{1}{(s/3)^2+1} = \frac{3}{s^2+9},$$

or use the Laplace transform for $\sin(3t)$ directly.

(ii) Notice that in some generality,

$$(\mathcal{L}f)(s) = \int_0^\infty \frac{f(t)}{t} te^{-st}\,\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \frac{f(t)}{t}e^{-st}\,\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathcal{L}\{\frac{f(\cdot)}{\cdot}\}(s)\right).$$

Integrating both sides in s gives us an expression for $\mathcal{L}\{\frac{f(\cdot)}{\cdot}\}(s)$.
Since $\mathcal{L}\{\sin(\cdot)\}(s) = 1/(s^2+1)$, we have

$$\mathcal{L}\{\sin(\cdot)/\cdot\}(s) = \int_s^\infty \frac{1}{u^2+1}\,\mathrm{d}u = \frac{\pi}{2} - \arctan(s) = \arctan(1/s).$$

(iii) Using the shifting theorem, we have

$$(\mathcal{L}f)(s) = \mathcal{L}\{\sin(\cdot)\}(s-1) = \frac{1}{(s-1)^2+1}.$$

(iv) Again, using the shifting theorem we have

$$(\mathcal{L}f)(s) = \mathcal{L}\{\cdot^2\}(s-1) = \frac{2}{(s-1)^3}.$$

4.

(i) The denominator is factorizable to

$$s^4 - 5s^2 + 4 = (x^2-1)(x^2-4) = (x-1)(x+1)(x-2)(x+2).$$

So the denominator has four distinct roots.
Using partial fractions we can write:

$$\frac{3s^2+6s+2}{s^4-5s^2+4} = \frac{-11/6}{s-1} + \frac{-1/6}{s+1} + \frac{13/6}{s-2} + \frac{-1/6}{s+2}.$$

Using the linearity of Laplace transforms and the shifting theorem, the inverse Laplace transform is

$$\frac{-11}{6}e^t - \frac{1}{6}e^{-t} + \frac{13}{6}e^{2t} + \frac{1}{6}e^{-2t}.$$

(ii) Set $f(t) = t\cos(\alpha t)$. Taking the derivative,

$$\begin{aligned} f'(t) &= \cos(\alpha t) - \alpha t\sin(\alpha t), \\ f''(t) &= -2\alpha\sin(\alpha t) - \alpha^2 t\cos(\alpha t). \end{aligned}$$

Then we can use the formula

$$(\mathcal{L}f'')(s) = s^2(\mathcal{L}f)(s) - sf(0) - f'(0)$$

to find that

$$2\alpha\mathcal{L}\{\sin(\alpha\cdot)\} + \alpha^2\mathcal{L}\{\cdot\cos(\alpha\cdot)\} = -s^2\mathcal{L}\{\cdot\cos(\alpha\cdot)\} + 1.$$

Collecting the terms we find

$$\mathcal{L}\{\cdot\cos(\alpha\cdot)\}(s^2+\alpha^2) = 1 - 2\alpha\mathcal{L}\{\sin(\alpha\cdot)\} = 1 - 2\alpha\frac{\alpha}{s^2+\alpha^2} = \frac{s^2-\alpha^2}{s^2+\alpha^2}.$$

Hence the inverse Laplace transform in question is

$$t\cos(\alpha t).$$

5. Taking the Laplace transform of the entire equation, we have

$$s^2\mathcal{L}y - sy(0) - y'(0) + 9\mathcal{L}y = \frac{8}{s+1}.$$

Putting in the initial conditions,

$$(s^2+9)\mathcal{L}y = \frac{8}{(s+1)}.$$

Re-arranging and using partial fractions,

$$\mathcal{L}y = \frac{8}{(s+1)(s^2+9)} = \frac{8/10}{(s+1)} + \frac{-8/10s}{s^2+9} + \frac{8/10}{s^2+9}.$$

Taking the inverse Laplace transform we arrive at:

$$y(t) = \frac{8}{10}e^{-t} - \frac{8}{10}\cos(3t) + \frac{8}{30}\sin(3t).$$

We can check that this satisfies the differential equation and the initial conditions.