TMA4130 Matematikk 4N Problem Sheet: 1st Week

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1.

(i) Use Taylor polynomials, or otherwise, to derive Euler's formula:

$$\exp(it) = \cos(t) + i\sin(t);$$

and use the formula to express $\sin(t)$ and $\cos(t)$ as a linear combination of two complex exponentials.

(ii) Using the linear combination derived show that

$$4\cos^3(\omega t) = \cos(3\omega t) + 3\cos(\omega t).$$

(iii) Derive the Laplace trasnform for $f(t) = \cos^3(\omega t)$.

(iv) What is the behaviour of $\mathcal{L}\{\sin(\omega\cdot)\}$ on $\mathbb{R}_+$ as $\omega \to \infty$?

2.

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with period $T > 0$ — i.e., $f(t) = f(t+T)$ for all $t \in \mathbb{R}$. Show that

$$(\mathcal{L}f)(s) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, \mathrm{d}t.$$

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with scaling symmetry given by $f(\alpha t) = \alpha^k f(t)$ for any $\alpha > 0$. Show that if $\mathcal{L}f$ exists, then

$$(\mathcal{L}f)(s) = \alpha^{-k-1}(\mathcal{L}f)(s/\alpha).$$

(iii) For f as in (ii), what can be said about the integral

$$\int_0^\infty f(t)\, \mathrm{d}t?$$

(iv) For f as in (ii), what is the behaviour of $\mathcal{L}f$ as $s \to \infty$?

(v) Suppose f is integrable and $g(t) = \int_0^t f(r)\, \mathrm{d}r$ is the antiderivative of f. Find $\mathcal{L}g$ in terms of $\mathcal{L}f$.

3. Compute the Laplace transforms of the following functions:

(i) $f(t) = \cos(3t + 3\pi/2)$,

(ii) $f(t) = \sin(t)/t$,

(iii) $f(t) = e^t \sin(t)$,

(iv) $f(t) = t^2 e^t$.

4. Find the inverse Laplace transform of the following functions:

(i)

$$\frac{3s^2 + 6s + 2}{s^4 - 5s^2 + 4},$$

(ii)

$$\frac{s^2 - \alpha^2}{(s^2 + \alpha^2)^2}.$$

5. Solve the following initial value problem using the Laplace transform:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 9y = 8e^{-t}, \qquad y(0) = \frac{\mathrm{d}y}{\mathrm{d}t}(0) = 0.$$