Avsn 11.9

F-transformen

A

tvenn

F-transform

$F(f)=\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ixw} dw$ (def)

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(w) e^{ixw} dw$ (teorem)

$\mathcal{F}(f') = iw\,\mathcal{F}(f)$

$\mathcal{F}(e^{-ax^2}) = ?$ $\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ax^2} e^{-ixw} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ax^2-ixw} dx$

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$\frac{d}{dw}\hat{f}(w) = \frac{d}{dw} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ax^2} e^{-ixw} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ax^2}(-ix) e^{-ixw} dx$

$= -\frac{i}{\sqrt{2\pi}} \int \underbrace{xe^{-ax^2}} e^{-ixw} dx = -i\,\mathcal{F}(\underbrace{xe^{-ax^2}})$

$= -i\,\mathcal{F}\left(-\frac{1}{2a}\left(e^{-ax^2}\right)'\right) = \frac{i}{2a}\mathcal{F}\left(\left(e^{-ax^2}\right)'\right) = \frac{i}{2a} iw\,\mathcal{F}(e^{-ax^2})$

$= -\frac{w}{2a}\hat{f}$

Def. gir: $\int \frac{d\hat{f}}{\hat{f}} = -\int \frac{w}{2a} dw \Rightarrow$ $\ln \hat{f} = -\frac{w^2}{4a} + \ln C$

Husk $\hat{f}(x) = e^{-ax^2}$

Videre $\hat{f}(w) = C e^{-w^2/4a}$

$C = \hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int e^{-ax^2} e^{-ixw} dx \Big|_{w=0} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ax^2} dx$ $z = \sqrt{a}\,x$, $dz = \sqrt{a}\,dx$

$= \frac{1}{\sqrt{2\pi a}} \int_{\mathbb{R}} e^{-z^2} dz = \frac{1}{\sqrt{2a}}$ fordi $\int_{\mathbb{R}} e^{-z^2} dz = \sqrt{\pi}$

Vi finner $\hat{f}(w) = \frac{1}{\sqrt{2a}} e^{-w^2/4a}$ for $f(x) = e^{-ax^2}$ $f = \hat{f}$ for $a = 1/2$

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Hvorfor $\int_{\mathbb{R}} e^{-z^2} dz = \sqrt{\pi}$

$\left(\int_{\mathbb{R}} e^{-z^2} dz\right)^2 = \left(\int e^{-\underset{x}{z}^2} dz\right)\left(\int e^{-\underset{y}{z}^2} dz\right) = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dx\,dy$

polar koord. $r^2 = x^2+y^2$, $dx\,dy = r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta = 2\pi \int_0^{\infty} r e^{-r^2} dr$

$= 2\pi \left[\left(-\frac{1}{2}\right) e^{-r^2}\right]_0^{\infty} = \pi$

Konvolusjon

Laplace: $y(t) = \int_0^t q(t-\tau) r(\tau) d\tau = q * r$

$\mathcal{L}(q * r) = \mathcal{L}(q)\,\mathcal{L}(r)$

Fourier: $f(x) = g * h(x) = \int_{-\infty}^{\infty} g(x-y) h(y) dy$

$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ixw} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}}\int_{\mathbb{R}} g(x-y) h(y) e^{-ixw} dy\,dx$

$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}}\int_{\mathbb{R}} h(y)\, g(x-y)\, e^{-ixw} dx\,dy$

$e^{-i(x-y)w} e^{-iyw}$

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$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}}\int_{\mathbb{R}} h(y)\, g(x-y)\, e^{-i(x-y)w} e^{-iyw} dy\,dx$

$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(y) e^{-iyw} \underbrace{\int_{\mathbb{R}} g(x-y) e^{-i(x-y)w} dx}_{z=x-y} dy$

$= \frac{1}{\sqrt{2\pi}} \underbrace{\int_{\mathbb{R}} h(y) e^{-iyw}} \left(\int_{\mathbb{R}} g(z) e^{-izw} dz\right) dy$ $\sqrt{2\pi}$ $1/\sqrt{2\pi}$

$= \sqrt{2\pi}\,\mathcal{F}(h)\,\mathcal{F}(g)$

Dvs $\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g)$ $\widehat{f * g} = \sqrt{2\pi}\,\hat{f}\,\hat{g}$

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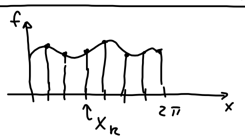
Diskret Fourier-transformasjon



$x_k = \frac{2\pi}{N} k$ $k = 0, \dots, N-1$

Vi kjenner $f(x_k) = f_k$, $k = 0, \dots, N-1$

Vi ønsker å skrive $f_k = \sum_{n=0}^{N-1} c_n e^{inx_k}$ ← Multipliser med e^{-inx_k}

Hvordan finner vi c_n?

Husk: $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

der $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) dx$

$\sum_{k=0}^{N-1} f_k e^{-imx_k} = \sum_{k=0}^{N-1}\left(\sum_{n=0}^{N-1} c_n e^{inx_k}\right) e^{-imx_k}$

$= \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_n e^{i(n-m)x_k}$

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$= \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} e^{i(n-m)x_k} = \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} \left(e^{i(n-m)\frac{2\pi}{N}}\right)^k$ $= r$

$= \sum_{n=0}^{N-1} c_n \underbrace{\sum_{k=0}^{N-1} r^k}$ der $r = e^{i(n-m)\frac{2\pi}{N}}$

$n = m \Rightarrow r = 1 \Rightarrow \sum_{k=0}^{N-1} r^k = N$

$n \neq m \Rightarrow$ ($r \neq 1$) $\sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r} = 0$ siden $r^N = e^{i(n-m)2\pi} = 1$

$= c_m N$

dvs $c_m = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-imx_k}$

Husk: $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

Gitt $(f_0, \dots, f_{N-1})$. Definerer $(\hat{f}_0, \dots, \hat{f}_{N-1})$

ved
$$\begin{cases} \hat{f}_m = \sum_{k=0}^{N-1} f_k e^{-imx_k} = N c_m \\ f_k = \frac{1}{N}\sum_{n=0}^{N-1} \hat{f}_n e^{inx_k} \end{cases}$$
Da er

Cooley-Tukey (1965) FFT Effektiv måte å regne DFT

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DFT:

$$f_k = \frac{1}{N}\sum_{n=0}^{N-1} \hat{f}_n e^{inx_k}$$

$$\hat{f}_m = \sum_{k=0}^{N-1} f_k e^{-imx_k}$$

Vi kan skrive $f = (f_0, \dots, f_{N-1})$, $\hat{f} = (\hat{f}_0, \dots, \hat{f}_{N-1})$

og dermed $\hat{f} = F_N f$

Påstand: $F_N = (e_{nk})$ der $e_{nk} = e^{-\frac{ink2\pi}{N}} = w^{nk}$

$$F_N = (e_{nk}) = \begin{bmatrix} e_{00} & e_{01} & \dots & e_{0,N-1} \\ e_{10} & & & \\ \vdots & & & \\ e_{N-1,0} & & & e_{N-1,N-1} \end{bmatrix} \text{ der } w = e^{-\frac{2\pi i}{N}}$$

$$F_N f = \begin{bmatrix} e_{00} & & e_{0,N-1} \\ \vdots & & \\ e_{N-1,0} & & e_{N-1,N-1} \end{bmatrix} \begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix} = \left(\sum_{k=0}^{N-1} e_{nk} f_k \right) = \left(\sum_{k=0}^{N-1} f_k e^{-in\overset{x_k}{\overset{\shortparallel}{\frac{2\pi k}{N}}}} \right) = (\hat{f}_n)$$

Dvs $\hat{f} = F_N f$ og dermed $\underline{f = F_N^{-1} \hat{f}}$

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Vi kan finne F_N^{-1} lett.

Vi har $F_N = (e_{nk})$ der $e_{nk} = e^{-\frac{2\pi ink}{N}} = w^{nk}$

der $w = e^{-2\pi i/N}$

Påstand $\boxed{\bar{F}_N F_N = NI}$

↑ kompleks-konjugerte til F_N

Vi har. $\bar{F}_N = (\bar{e}_{nk}) = (\bar{w}^{nk})$

$$\bar{F}_N F_N = \begin{pmatrix} \bar{e}_{00} & \dots & \bar{e}_{0,N-1} \\ \vdots & & \\ \bar{e}_{N-1,0} & & \bar{e}_{N-1,N-1} \end{pmatrix} \begin{pmatrix} e_{00} & e_{0,N-1} \\ \vdots & \\ e_{N-1,0} & e_{N-1,N-1} \end{pmatrix}$$

(n,m) element

$$\sum_{k=0}^{N-1} \bar{e}_{nk} e_{km} = \sum_{k=0}^{N-1} e^{\frac{2\pi ink}{N}} e^{-\frac{2\pi imk}{N}} = \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}(n-m)k} = \begin{cases} 0 & n \neq m \\ N & n = m \end{cases}$$

Dermed $F_N^{-1} = \frac{1}{N} \bar{F}_N$!

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F-analyse

F-rekker $f(x) = a_0 + \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2}(\cancel{f(x+0)} + f(x-0))$

der $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$

Kompleks form: $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx}$

$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{+inx} dx$

Kunne skalere til vilkårlig periode 2L

F-transformasjonen: $\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ixw} dx$

$\frac{1}{2}(f(x+0) + f(x-0)) = f(x) = \frac{1}{\sqrt{2\pi}} \int \hat{f}(w) e^{ixw} dw$

$\widehat{f'} = iw\hat{f}$

$\widehat{f * g} = \sqrt{2\pi}\, \hat{f}\, \hat{g}$

$(e^{-ax^2})^\wedge$

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