16.9.19

Fourier-rekker

$$f = a_0 + \sum (a_n \cos nx + b_n \sin nx)$$

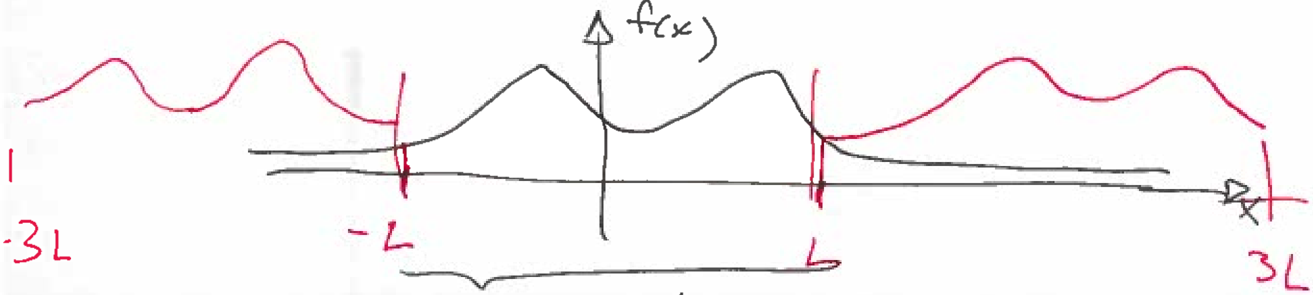
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Kompleks form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{der} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

$$a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n})$$

Fourier-transformasjonen

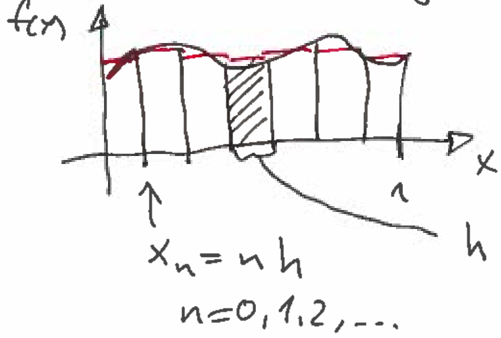


F-rekken $\quad L \to \infty$

$$f_L(x) = f(x) \quad \text{på } [-L, L]$$

$$f_L(x+2L) = f_L(x)$$

Riemann-integral



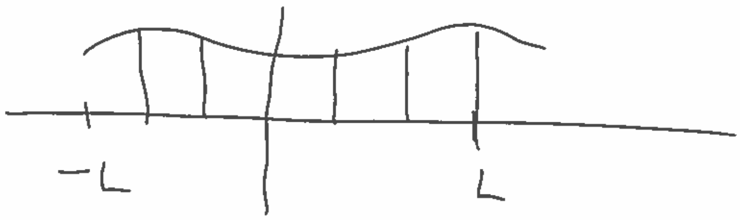
$$\int_0^1 f(x)\,dx = \lim_{h\to 0} \sum_{n=0}^{1/h} \underbrace{f(x_n)h}_{\text{areal av et rektangel}}$$

F-rekken til f_L:

$$f_L(x) = \sum c_n e^{i\frac{n\pi}{L}x} \quad \text{der } c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\frac{\pi}{L}x}\,dx$$

$$f_L(x) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2L}\int_{-L}^{L} f(z)\, e^{-in\frac{\pi}{L}z}\,dz\right) e^{in\frac{\pi}{L}x}$$

$$= \sum \int_{-L}^{L} f(z)\, e^{-in\frac{\pi}{L}z}\,dz\; e^{i\overbrace{n\frac{\pi}{L}}^{\to x_n}x}\;\frac{\pi}{2\pi L}$$



$$= \frac{1}{2\pi}\sum_n \underbrace{\int_{-L}^{L} f(z)\, e^{-in\frac{\pi}{L}z}\,dz\; e^{i\overbrace{n\frac{\pi}{L}}^{\text{"}\cancel{x}_n\text{"}}x}}_{\text{"}f\text{"}}\;\overbrace{\frac{\pi}{L}}^{\text{"}h\text{"}}$$

$$\xrightarrow[L\to\infty]{} \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)\, e^{-i\omega z}\,dz\; e^{i\omega x}\,d\omega$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)\, e^{-i\omega z}\,dz\right)}_{\hat{f}(\omega)} e^{i\omega x}\,d\omega = f(x)$$

F-transformasjonen:

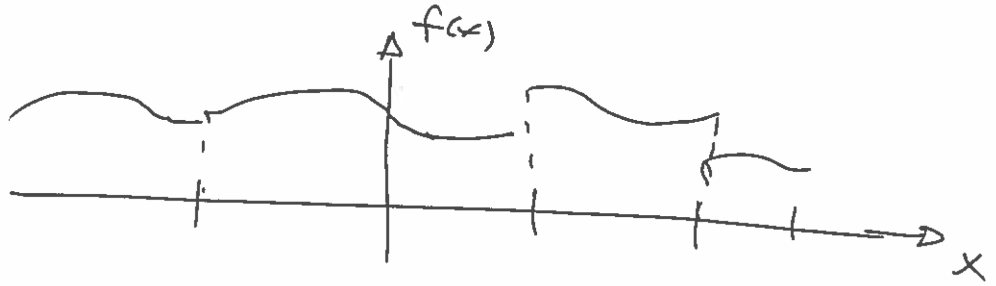
$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\,dx = \mathcal{F}(f)$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\,d\omega = \mathcal{F}^{-1}(\hat{f})$$

Teorem Anta at $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$

Anta at f er kontinuerlig deriverbar untatt i et endelig antall punkter, og anta at $\int_{-\infty}^{\infty} |f'(x)|\,dx < \infty$

Da vil $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx}\,dw = \frac{1}{2}\left(f(x+0) + f(x-0)\right)$



Eks

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (\cos wx + i \sin wx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos wx\,dx = \frac{1}{\sqrt{2\pi}} \Big|_{-1}^{1} \frac{\sin wx}{w}$$

$$= \frac{2}{\sqrt{2\pi}} \frac{\sin w}{w} = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}$$

Vi får:

$$\frac{1}{2}\left(f(x+0) + f(x-0)\right) = \frac{1}{\sqrt{2\pi}} \int \hat{f}(w) e^{iwx}\,dw =$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin w}{w} e^{iwx}\, dw$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{\frac{\sin w}{w}}_{\text{like}} \left(\underbrace{\cos wx}_{\text{like}} + i \underbrace{\sin wx}_{\text{odde}} \right) dw$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin w}{w} \cos wx \, dw$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin w}{w} \cos wx \, dw = f(x) = \begin{cases} 1, & |x| < 1 \\ 1/2 & |x| = 1 \\ 0, & |x| > 1 \end{cases}$$

dvs

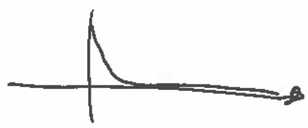
$$\int_0^{\infty} \frac{\sin w}{w} \cos wx \, dw = \begin{cases} \pi/2 & |x| < 1 \\ \pi/4 & |x| = 1 \\ 0 & |x| > 1 \end{cases}$$

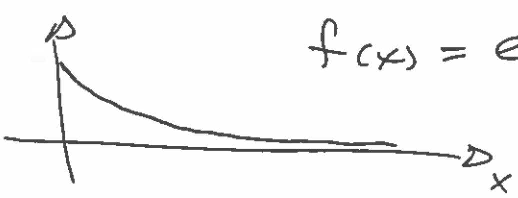
$x=0$: $$\int_0^{\infty} \frac{\sin w}{w} \, dw = \pi/2$$

$$\int_0^{x} \frac{\sin w}{w} \, dw = \mathrm{Si}(x)$$

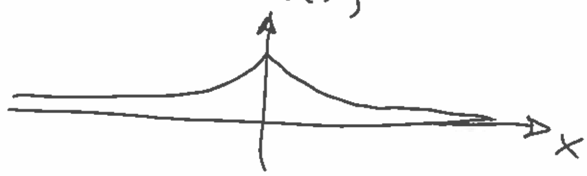
Eks Laplace-integraler

$$f(x) = e^{-kx}, \quad x > 0, \quad k > 0$$





(1) Like utvidelse



$$f(x) = e^{-k|x|}$$

$$\hat{f}(w) = ?$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left(\cos wx + i \sin wx \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{like}} \left(\underbrace{\cos \omega x}_{\text{like}} + i \underbrace{\sin \omega x}_{\text{odde}} \right) dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-kx} \cos \omega x \, dx = \underline{\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-kx} \cos \omega x \, dx}$$

$$= \sqrt{\frac{2}{\pi}} \left. \left(-\frac{1}{k}\right) e^{-kx} \cos \omega x \right|_{0}^{\infty} - \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \left(-\frac{1}{k}\right) e^{-kx} (-\omega) \sin \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{k} - \sqrt{\frac{2}{\pi}} \frac{\omega}{k} \int_{0}^{\infty} e^{-kx} \sin \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{k} - \sqrt{\frac{2}{\pi}} \frac{\omega}{k} \left(\left. \left(-\frac{1}{k}\right) e^{-kx} \sin \omega x \right|_{0}^{\infty} - \int_{0}^{\infty} \left(-\frac{1}{k}\right) e^{-kx} \omega \cos \omega x \, dx \right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{k} - \sqrt{\frac{2}{\pi}} \left(\frac{\omega}{k}\right)^2 \underline{\int_{0}^{\infty} e^{-kx} \cos \omega x \, dx}$$

dvs

$$\sqrt{\frac{2}{\pi}} \left(1 + \left(\frac{\omega}{k}\right)^2\right) \int_{0}^{\infty} e^{-kx} \cos \omega x \, dx = \sqrt{\frac{2}{\pi}} \frac{1}{k}$$

eller:

$$\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-kx} \cos \omega x \, dx = \sqrt{\frac{2}{\pi}} \frac{1}{k} \frac{1}{1 + \omega^2/k^2}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{k} \frac{k^2}{k^2 + \omega^2}$$

$$= \sqrt{\frac{2}{\pi}} \frac{k}{k^2 + \omega^2}$$

dvs

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{k}{k^2 + \omega^2}$$

Vi får da:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega = \frac{1}{\cancel{\sqrt{2}}\sqrt{\pi}} \sqrt{\frac{\cancel{2}}{\pi}} \int_{-\infty}^{\infty} \frac{k}{k^2 + \omega^2} e^{i\omega x} d\omega$$

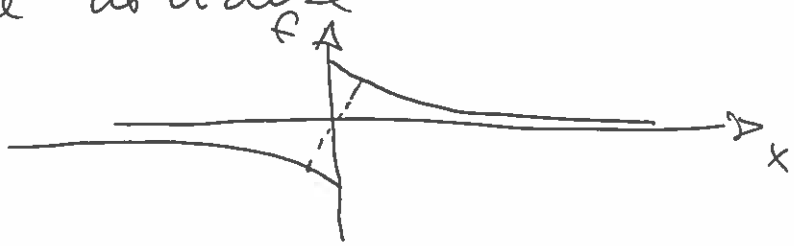
$$= f(x) = e^{-k|x|}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{k}{k^2+\omega^2} (\cos \omega x + i \sin \omega x) d\omega$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{k}{k^2+\omega^2} \cos \omega x \, d\omega = e^{-k|x|}$$

$x=0$ $$\int_0^{\infty} \frac{k\,d\omega}{k^2+\omega^2} = \frac{\pi}{2}$$

(2) Odde utvidelse



$\hat{f}(\omega) = ?$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (\cos \omega x + i \sin \omega x) dx$$

$$= \sqrt{\frac{2}{\pi}}\, i \int_0^{\infty} f(x) \sin \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}}\, i \underbrace{\int_0^{\infty} e^{-kx} \sin \omega x \, dx}$$

$$= \sqrt{\frac{2}{\pi}}\, i \left(\Big|_0^{\infty} -\frac{1}{k} e^{-kx} \sin \omega x - \int_0^{\infty} \left(-\frac{1}{k}\right) e^{-kx} \omega \cos \omega x \, dx \right)$$

$$= \sqrt{\frac{2}{\pi}}\, i \frac{\omega}{k} \left[\Big|_0^{\infty} \left(-\frac{1}{k}\right) e^{-kx} \cos \omega x - \int_0^{\infty} \left(-\frac{1}{k}\right) e^{-kx} (-\omega) \sin \omega x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}}\, i \frac{\omega}{k} \left(\frac{1}{k} - \frac{\omega}{k} \underbrace{\int_0^{\infty} e^{-kx} \sin \omega x \, dx} \right)$$

$$\sqrt{\frac{2}{\pi}}\, i \left(1+\left(\frac{\omega}{h}\right)^2\right) \int_0^\infty e^{-hx}\sin\omega x\,dx = \sqrt{\frac{2}{\pi}}\, i\, \frac{\omega}{h^2}$$

$$\sqrt{\frac{2}{\pi}}\, i \int_0^\infty e^{-hx}\sin\omega x\,dx = \sqrt{\frac{2}{\pi}}\, i\, \frac{\omega}{h^2}\,\frac{1}{1+(\omega/h)^2}$$

$$= \sqrt{\frac{2}{\pi}}\, i\, \frac{\omega}{h^2+\omega^2}$$

dvs:

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}}\, i\, \frac{\omega}{h^2+\omega^2}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\,d\omega = \frac{1}{\cancel{\sqrt{2\pi}}}\,\sqrt{\frac{\cancel{2}}{\pi}}\, i \int_{-\infty}^{\infty} \frac{\omega}{h^2+\omega^2}\, e^{i\omega x}\,d\omega$$

$$= \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\omega}{h^2+\omega^2}\left(\cos\omega x + i\sin\omega x\right)d\omega$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{\omega}{h^2+\omega^2}\sin\omega x\,d\omega = e^{-hx}, \quad x>0$$

F-transformasjonen er lineær:

$$\mathcal{F}(c_1 f_1 + c_2 f_2) = c_1\mathcal{F}(f_1) + c_2\mathcal{F}(f_2)$$

$$\widehat{c_1 f_1 + c_2 f_2} = c_1 \hat{f}_1 + c_2 \hat{f}_2, \quad c_1, c_2 \in \mathbb{R}$$

$$\mathcal{F}(f') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\,e^{-i\omega x}\,dx \qquad \underbrace{e^{-i\omega x}dx}$$

$$= \frac{1}{\sqrt{2\pi}}\Big| f(x)\,e^{-i\omega x}\Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,(-i\omega)$$

$$= i\omega\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}\,dx = i\omega\,\mathcal{F}(f)$$

$$\boxed{\begin{aligned}\mathcal{F}(f') &= i\omega\,\mathcal{F}(f)\\ \widehat{f'} &= i\omega\,\hat{f}\end{aligned}}$$

$$\underline{\mathcal{F}(f'')} = i\omega\,\mathcal{F}(f') = (i\omega)^2\,\mathcal{F}(f) = \underline{-\omega^2\,\mathcal{F}(f)}$$