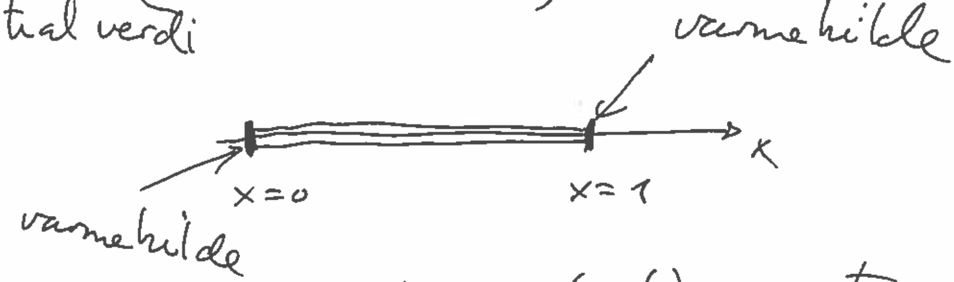
11.11.19

Numerisk løsning av partielle differensial-ligninger

Varmeledningsligningen med Dirichlet randbetingelser og initial verdi

$$
\begin{cases}
u_t = u_{xx} & x \in [0,1],\ t \geq 0 \\
u(0,t) = g_0(t) \\
u(1,t) = g_1(t) \\
u(x,0) = f(x)
\end{cases}
$$



$u = u(x,t)$ er temperaturen i pkt x ved tiden t.

Velg $M >> 1$. La $\Delta x = 1/M$ og $x_i = i\Delta x$ der $i = 0, \ldots, M$

Taylor-utvikler høyresiden av ligningen rundt punktet x_i:

$$
\frac{\partial u}{\partial t}(x_i,t) = \frac{1}{\Delta x^2}\left(u(x_{i+1},t) - 2\,u(x_i,t) + u(x_{i-1},t)\right) \cancel{+\mathcal{O}(\Delta x^2)}
$$

Dropper feil-leddet, og innfører

$$
v_i(t) \approx u(x_i,t)
$$

og får:

System av 1-ordens ordinære diff. lign.

$$
\begin{cases}
\frac{d}{dt} v_i(t) = \frac{1}{\Delta x^2}\left(v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)\right) \quad \text{for } i = 1, \ldots, M-1 \\
v_0(t) = g_0(t) \\
v_M(t) = g_1(t)
\end{cases}
$$

$i=1$: $\frac{d}{dt} v_1(t) = \frac{1}{\Delta x^2}\left(v_2(t) - 2v_1(t) + \overset{= g_0(t)}{v_0(t)} \right)$

$i=2$: $\frac{d}{dt} v_2(t) = \frac{1}{\Delta x^2}\left(v_3(t) - 2v_2(t) + v_1(t) \right)$

osv.

Vi kan skrive dette som

$$\frac{d}{dt} v(t) = F(t, v(t))$$

$$v(0) = f$$

der $v = \begin{pmatrix} v_1 \\ \vdots \\ v_{M-1} \end{pmatrix}$, $F(t, v(t)) = \frac{1}{\Delta x^2}\left(v_{i+1} - 2v_i + v_{i-1} \right)$

$$v(0) = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_{M-1}) \end{pmatrix}$$

Løser dette systemet ved hjelp av forlengs Euler: Velg $N \gg 1$ og $\Delta t = 1/N$, $t_n = n\Delta t$

La $v_i^n \approx u(x_i, t_n)$ (ofte $v_{i,n}$)

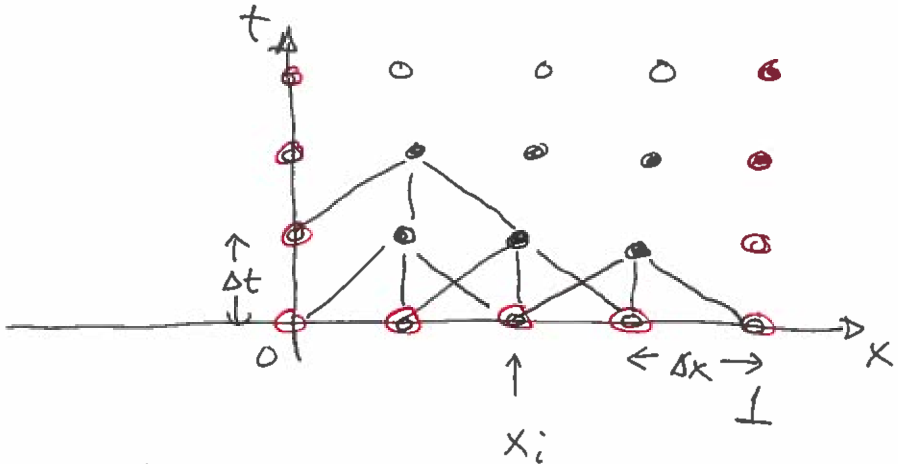
Da får vi:

$$v_i^{n+1} = v_i^n + \Delta t\, F(v^n)$$

$$= v_i^n + \frac{\Delta t}{\Delta x^2}\left(v_{i+1}^n - 2v_i^n + 2v_{i-1}^n \right)$$

der $i = 1, \ldots, M-1$ og $n = 0, 1, 2, \ldots$

Dette kan illustreres som følger:



o = kjent
o = ukjent

$$U_1^1 = U_1^0 + \frac{\Delta t}{\Delta x^2}\left(U_2^0 - 2U_1^0 + U_0^0\right)$$

$$U_2^1 = U_2^0 + \frac{\Delta t}{\Delta x^2}\left(U_3^0 - 2U_2^0 + U_1^0\right), \text{ osv}$$

Nå kjenner vi alle U_i^1, $i = 1, \dots, M-1$

Tilsvarende regner vi ut U_i^2, $i = 1, \dots, M-1$

Vi fortsetter slik og finner alle U_i^n!

Eks

$$u_t = u_{xx}$$
$$g_0 = g_1 = 0$$
$$f(x) = \sin(\pi x)$$

La nå $M = 4$, $N = 20$ slik at $\Delta x = 1/4$, $\Delta t = 1/20$

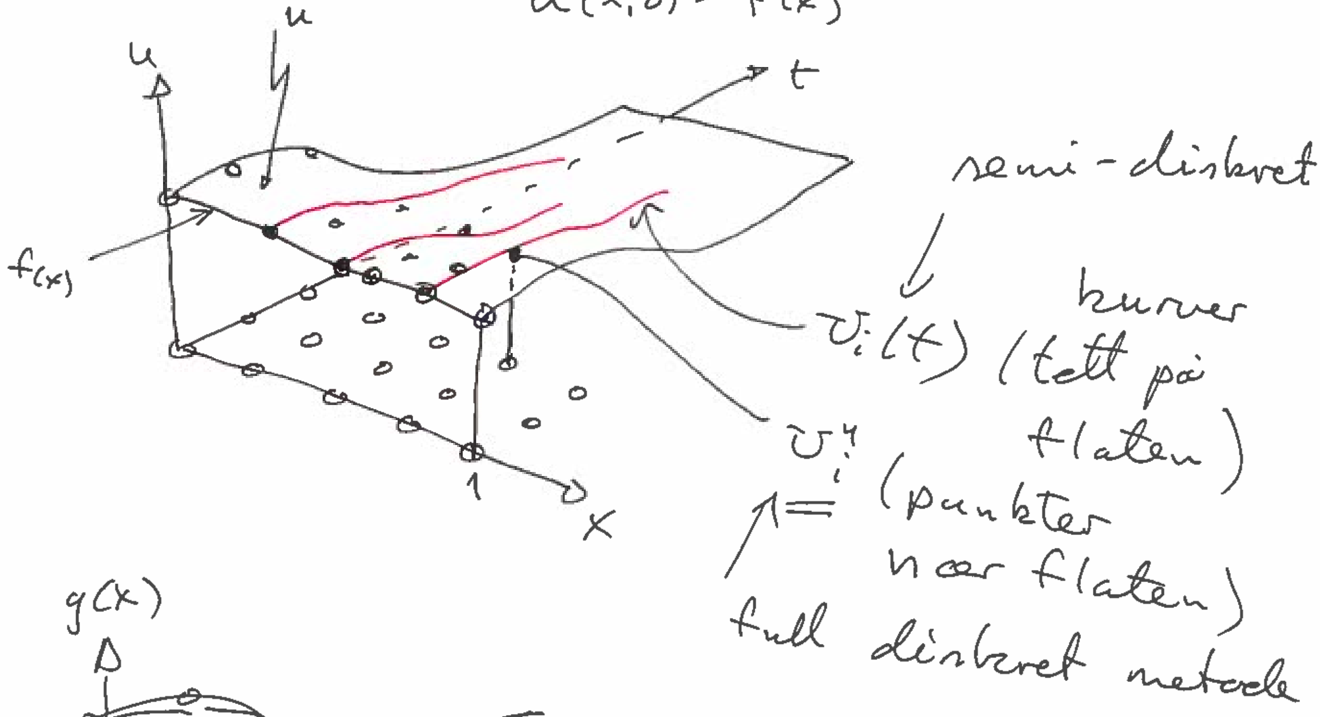
Da er:

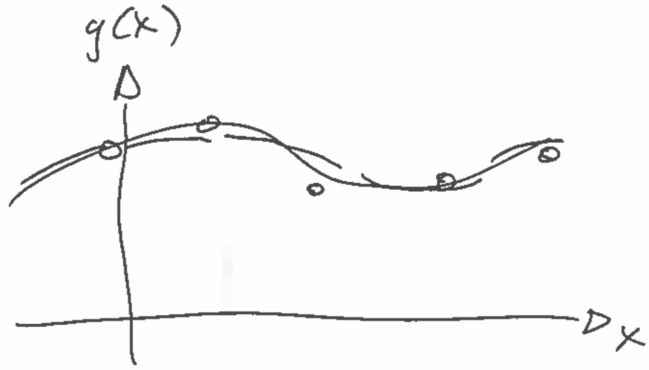
$$U_i^{n+1} = U_i^n + \frac{1/20}{(1/4)^2}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right)$$
$$= U_i^n + \frac{4}{5}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right)$$

$$U_i^0 = f(x_i) = \sin(\pi i \Delta x)$$

Ligningene kan løses eksplisitt.

Gitt $u_t = u_{xx}$, $u(0,t) = g_0(t)$, $u(1,t) = g_1(t)$
$u(x,0) = f(x)$





Vi hadde parameteren $r = \frac{\Delta t}{\Delta x^2}$

Stabilitetsanalyse vil gi en begrensning på r.

Vi har

$$\frac{d}{dt} v_i = \frac{1}{\Delta x^2}\left(v_{i+1} - 2v_i + v_{i-1}\right), \quad i = 1, \dots, M-1$$

$$v_0 = g_0, \quad v_M = g_1$$

som vil løse ligningssystemet:

$$\frac{d}{dt} v = \frac{1}{\Delta x^2}(A v + g)$$

der

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_{M-1} \end{pmatrix}, \quad g = \begin{pmatrix} g_0 \\ 0 \\ \vdots \\ 0 \\ g_1 \end{pmatrix} \quad \text{og}$$

$$A = \begin{pmatrix} -2 & +1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{pmatrix}$$

Eks:

$$\frac{d}{dt} v_1 = \frac{1}{\Delta x^2}\left[(-2v_1 + v_2) + g_0\right]$$
$$= \frac{1}{\Delta x^2}\left(v_2 - 2v_1 + v_0\right)$$

og
$$\frac{d}{dt} v_2 = \frac{1}{\Delta x^2}\left[v_1 - 2v_2 + v_3\right]$$
$$= \frac{1}{\Delta x^2}\left(v_3 - 2v_2 + v_1\right)$$

Dette har vi studert tidligere i forb. m. stive ligninger.

$$\frac{d}{dt} v = \frac{1}{\Delta x^2}(A + g)$$

Forlengs Euler betyr at vi må kreve

$$-2 \leq \Delta t \, \lambda_k \leq 0 \qquad k = 1, \ldots, M-1$$

der λ_k er egenverdiene til $\frac{1}{\Delta x^2} A$ for å få stabilitet.

Egenverdiene til A er μ_k og er gitt ved

$$\mu_k = -4 \sin^2\left(\frac{k\pi}{M}\right), \quad k = 1, \ldots, M-1$$

Vi har at $\quad -4 \leq \mu_k \leq 0$

Egenverdiene til $\frac{1}{\Delta x^2} A$ blir da

$$\lambda_k = \frac{1}{\Delta x^2} \mu_k$$

Vi har:

$$-4/\Delta x^2 \leq \frac{1}{\Delta x^2} \mu_k = \lambda_k \leq 0$$

eller

$$-\frac{4\Delta t}{\Delta x^2} \leq \Delta t \lambda_k \leq 0$$

Stabilitet krever at

$$-2 \leq -4 \frac{\Delta t}{\Delta x^2}$$

eller:

$$\underline{r = \frac{\Delta t}{\Delta x^2} \leq 1/2}$$

Vi ønsker at $\Delta x, \Delta t \to 0$, men stabilitet krever at $\underline{\underline{r = \frac{\Delta t}{\Delta x^2} \leq 1/2}}$

Baklengs Euler gir en feil på $O(\Delta t + \Delta x^2)$ i hvert gridpkt.

Eksamen, høst 2010, oppg 9

Vi har $t_j = jk$ og $x_i = ih$ der $h = 1/N$

Innfør $v_i^j \approx u(x_i, t_j)$ for den tilnærmede løsningen.

Sentraldifferanse:

$$u_{xx}(x_i, t_j) \approx \frac{1}{h^2}\big(u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)\big)$$

$$\approx \frac{1}{h^2}\left(v_{i+1}^j - 2v_i^j + v_{i-1}^j\right)$$

Forlengs Euler:

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{1}{k}\big(u(x_i, t_{j+1}) - u(x_i, t_j)\big)$$

$$\approx \frac{1}{k}\left(v_i^{j+1} - v_i^j\right)$$

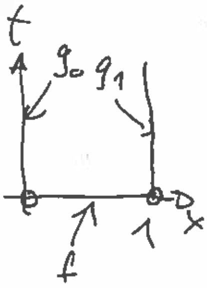
Innsatt i ligningen: $u_t = \kappa u_{xx} + x(1-x)$ får vi:

$$\frac{1}{k}\left(v_i^{j+1} - v_i^j\right) = \frac{\kappa}{h^2}\left(v_{i+1}^j - 2v_i^j + v_{i-1}^j\right) + x_i(1-x_i)$$

eller:

$$v_i^{j+1} = v_i^j + \frac{\kappa k}{h^2}\left(v_{i+1}^j - 2v_i^j + v_{i-1}^j\right) + k\,x_i(1-x_i)$$

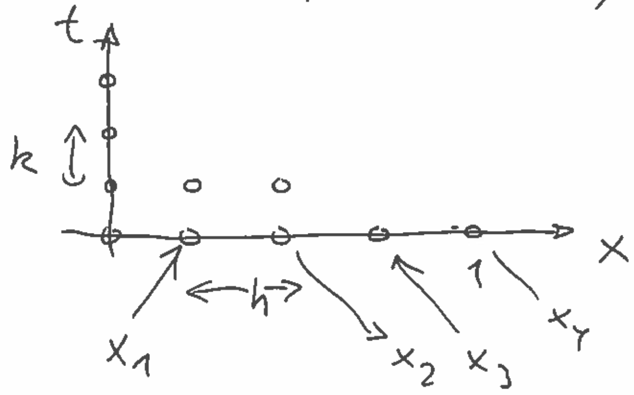
$i = 1, \dots, N-1$



$$v_0^j = v_N^j = 0$$

$$v_i^0 = \sin(\pi x_i), \quad i = 1, \dots, N-1$$

La $\kappa=0.1$, $h=0.25$, $h=0.2$



Vi har at $u(0.25, 0.2) \approx U_1^1$

og $u(0.5, 0.2) \approx U_2^1$

Differanseskjemaet gir:

$$U_i^{j+1} = U_i^j + \frac{\kappa k}{h^2}\left(U_{i+1}^j - 2U_i^j + U_{i-1}^j\right) + k x_i (1-x_i)$$

videre:

$$U_1^1 = U_1^0 + \frac{0.2\cdot 0.2}{(0.25)^2}\left(U_2^0 - 2U_1^0 + U_0^0\right) + h x_1(1-x_1)$$

$$= \sin(\pi x_1) + \frac{0.2\cdot0.2}{(0.25)^2}\left(\sin(\pi x_2) - 2\sin(\pi x_1) + \sin(\pi x_0)\right) + 0.2\, x_1(1-x_1)$$

$$= \sin(\pi/4) + \frac{0.2\cdot0.2}{(0.25)^2}\left(\sin(\pi/2) - 2\sin(\pi/4) + \sin(0)\right) + 0.2\cdot 0.25(1-0.25)$$

$$= \dots = 0.6121$$

Tilsvarende for U_2^1.