14.10.19

Polynominterpolasjon

Tilnærme gitt funksjon $f(x)$ med polynomer p_n av grad n slik at

$$f(x_j) = p_n(x_j) \quad \text{for } j = 0, \dots, n$$

der $x_0, \dots, x_n$ er $n+1$ gitte punkter.

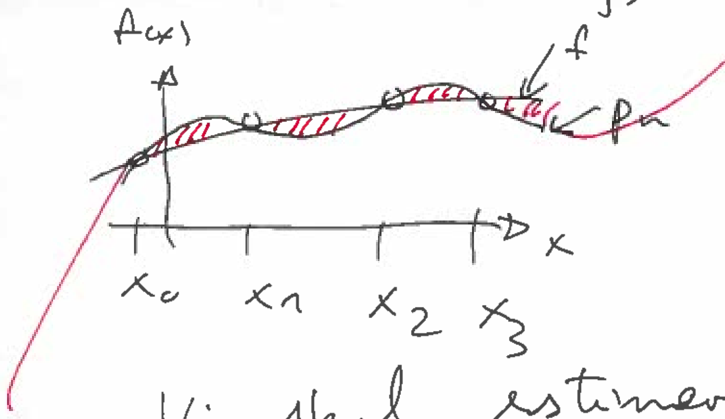
To eksempler: Lagrange
Newton

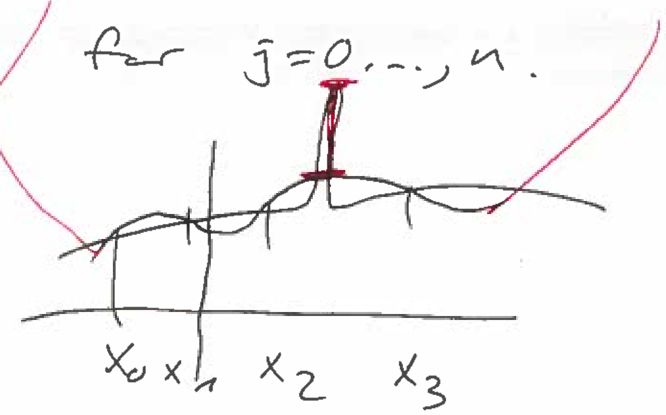
Det fins <u>entydig</u> slikt polynom p_n.

Feilestimater

$$e(x) = f(x) - p_n(x)$$

Vi har $e(x_j) = 0$ for $j = 0, \dots, n$.





Vi skal estimere $e(x) = f(x) - p_n(x)$ og finne den største verdien av $e(x)$.

Ser på et intervall $[a,b]$.

Gitt et pkt $x \in [a,b]$

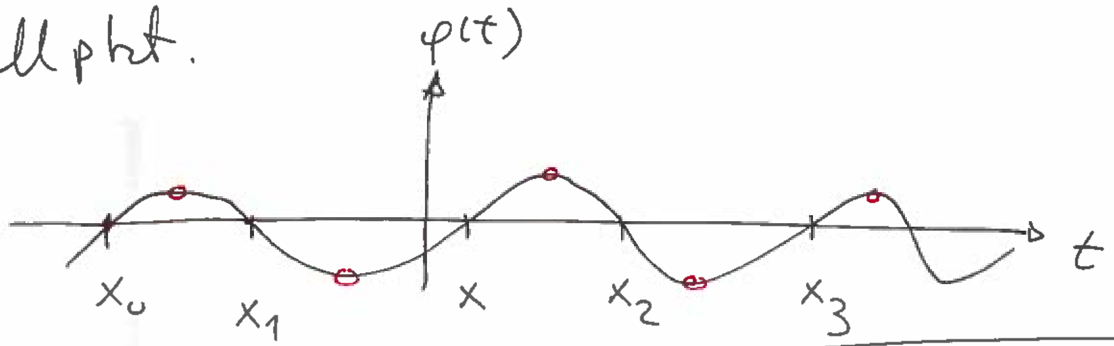
Definer funksjonen $\varphi(t) = e(t)\,\omega(x) - e(x)\,\omega(t)$

der $\omega(x) = \prod_{j=0}^{n} (x - x_j) = (x-x_0)(x-x_1)\dots(x-x_n)$

Da er $\varphi(x_j) = \overset{=0}{e(x_j)}\,\omega(x) - e(x)\,\overset{=0}{\omega(x_j)} = 0$

$\varphi(x) = e(x)\,\omega(x) - e(x)\,\omega(x) = 0$

Det betyr at $\varphi(t)$ har minst $n+2$ nullpkt.



Rolles teorem: La φ være en kontinuerlig deriverbar funksjon. Anta $\varphi(a) = \varphi(b) = 0$. Da fins $z \in (a,b)$ slik at $\varphi'(z) = 0$

Det betyr at $\varphi'(t)$ har minst $n+1$ nullpkt.

Videre har $\varphi''(t)$ minst n nullpkt.

Det gir tilslutt at $\varphi^{(n+1)}(t)$ har minst ett nullpkt. Vi kaller det nullpkt for $\xi(x)$.

Vi antar her at f er minst $n+1$ ganger deriverbar.

Vi har at $\varphi(t) = e(t)\,\omega(x) - e(x)\,\omega(t)$

$e(t) = f(t) - p_n(t)$. Vi har at $e^{(n+1)}(t) = f^{(n+1)}(t)$

og $\omega^{(n+1)}(t) = (n+1)!$

Dvs:

$$\varphi^{(n+1)}(t) = f^{(n+1)}(t)\,\omega(x) - e(x)\,(n+1)!$$

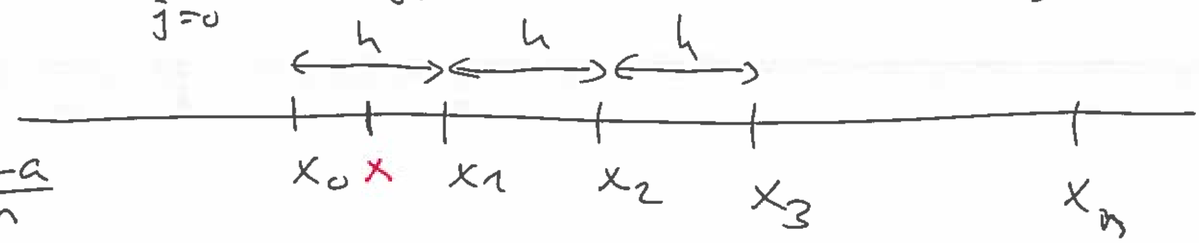
Sett inn $t = \xi(x)$. Da får vi:

$$0 = \varphi^{(n+1)}(\xi(x)) = f^{(n+1)}(\xi(x))\,\omega(x) - e(x)\,(n+1)!$$

som gir:

$$e(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}\,\cancel{\omega(x)} \prod_{j=0}^{n}(x - x_j)$$

$$\omega(x) = \prod_{j=0}^{n}(x - x_j) = (x - x_0)(x - x_1)(x - x_2)\ldots(x - x_n)$$



$h = \frac{b-a}{n}$

$$|\omega(x)| \le h\,h\,2h\,3h \ldots nh = h^{n+1}\,n!$$

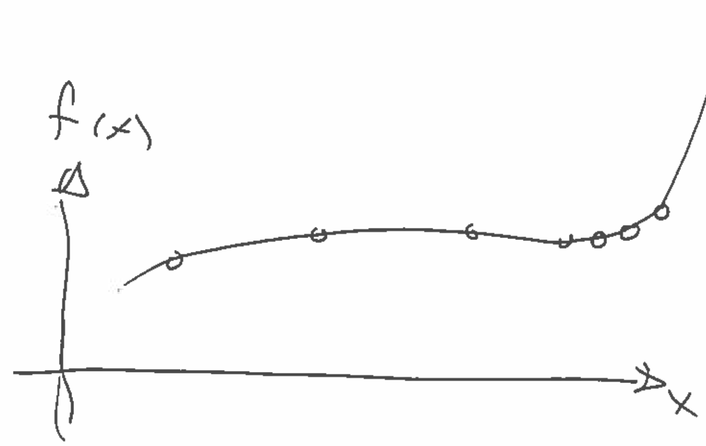
Vi ser at $|\omega(x)| \le h^{n+1}\,n!$

Man kan vise at $|\omega(x)| \le \frac{1}{4} h^{n+1}\,n!$

Det gir at

$$|e(x)| \le \frac{1}{4}\,\frac{h^{n+1}}{n+1}\,M$$

der $M = \max_{x \in [a,b]} |f^{(n+1)}(x)|$



Hva er optimalt valg av x_j?

Det betyr å velge x_j slik at

$\omega(x) = \prod (x - x_j)$ blir minst mulig.

Man kan vise at de såkalte Chebyshev-punktene:

$$\tilde{x}_j = \cos\left(\frac{(2j+1)\pi}{2(n+1)}\right)$$

for $j = 0, \dots, n$

gir den minste $|\omega(x)|$.

Vi skriver

$$\omega_{Cheb}(x) = \prod_{j=0}^{n} (x - \tilde{x}_j)$$

og man kan vise at

$$\frac{1}{2^n} = \max_{x \in [-1,1]} |\omega_{Cheb}(x)| \le \max_{x \in [-1,1]} |q(x)|$$

for et vilkårlig polynom q av grad $n+1$.

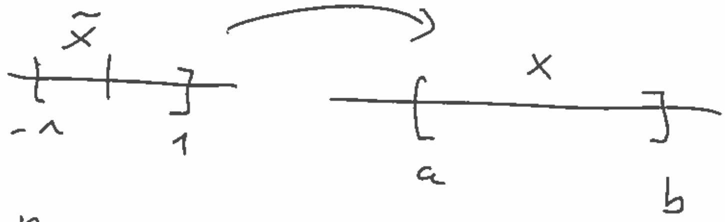
Vi kan skalere dette resultatet slik at gjelder for et vilkårlig intervall.

Vilkårlig intervall:

$$x = \frac{b-a}{2}\tilde{x} + \frac{b+a}{2}$$

$\tilde{x} = -1 : \quad x = a$

$\tilde{x} = 1 : \quad x = b$



$$w(x) = \prod_{j=0}^{n}(x - x_j) = \prod_{j=0}^{n}\left(\frac{b-a}{2}\tilde{x} + \cancel{\frac{b+a}{2}} - \left(\frac{b-a}{2}\tilde{x}_j + \cancel{\frac{b+a}{2}}\right)\right)$$

$$= \left(\frac{b-a}{2}\right)^{n+1} \prod(\tilde{x} - \tilde{x}_j) = \left(\frac{b-a}{2}\right)^{n+1} w_{cheb}(\tilde{x})$$

som gir:

$$|w(x)| \leq \left(\frac{b-a}{2}\right)^{n+1} |w_{cheb}(\tilde{x})|$$

$$= \left(\frac{b-a}{2}\right)^{n+1} \frac{1}{2^n}$$

Dermed

$$\max_{x\in[a,b]} |f(x) - p_n(x)| \leq \frac{(b-a)^{n+1}}{2^{2n+1}(n+1)!} M$$

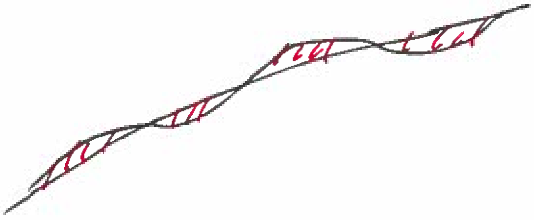
der $M = \max_{x\in[a,b]} |f^{(n+1)}(x)|$

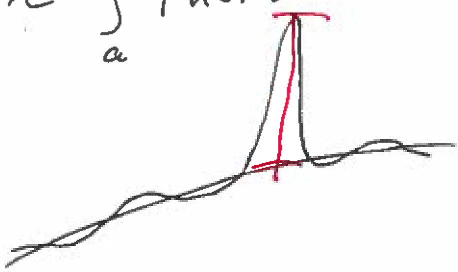
Numerisk integrasjon

$$I = \int_a^b f(x)dx = ?$$

Naturlig å bruke tilnærming med polynomer. Om $f(x) \approx p_n(x)$, bør også

$$\int_a^b f(x)dx \approx \int_a^b p_n(x)dx$$





Gitt funksjon f og polynom av grad n slik at $f(x_j) = p_n(x_j)$, $j = 0, \dots, n$ for gitte punkter $x_0, \dots, x_n$.

Lagrange polynomet er gitt ved:

$$p_n(x) = \sum_{j=0}^{n} f(x_j)\, l_j(x)$$

der

$$l_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)}$$

Ideen er følgende:

$$\int_a^b f(x)dx \approx \int_a^b p_n(x)dx$$

$$= \sum_{j=0}^{n} f(x_j) \underbrace{\int_a^b l_j(x)dx}_{w_j}$$

$$= \sum_{j=0}^{n} w_j f(x_j)$$

Kvadratur

$$I(f)=\int_a^b f(x)\,dx \approx \underbrace{\sum_{j=0}^{n} w_j f(x_j)}_{=Q(f)} \quad \text{der } w_j=\int_a^b l_j(x)\,dx$$

<u>Eks</u>

$$I(f)=\int_0^1 \underbrace{\cos\left(\frac{\pi x}{2}\right)}_{=f(x)}dx = \frac{2}{\pi} \approx 0.636..$$

<u>Alt 1</u> $x_0=0,\ x_1=1$

$$l_0=\frac{x-x_1}{x_0-x_1}=\frac{x-1}{-1}=1-x \quad , w_0=1/2$$

$$l_1=\frac{x-x_0}{x_1-x_0}=\frac{x}{1}=x \quad , w_1=1/2$$

$$Q(f)=\frac{1}{2}\left(f(0)+f(1)\right)=1/2=0.50$$

<u>Alt 2</u> $x_0=\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right),\ x_1=\frac{1}{2}\left(1-\frac{\sqrt{3}}{3}\right)$

$$l_0=\frac{x-x_1}{x_0-x_1}=-\sqrt{3}x+\frac{1}{2}(1+\sqrt{3}), \quad w_0=1/2$$

$$l_1=\frac{x-x_0}{x_1-x_0}=\sqrt{3}x+\frac{1}{2}(1-\sqrt{3}), \quad w_1=1/2$$

$$Q(f)=\frac{1}{2}\left(f(x_0)+f(x_1)\right)=0.636...$$

Her er $|Q(f)-I(f)|=9.7\cdot 10^{-4}$

Om f er et polynom av grad $\leq n$, så er $f(x)=p_n(x)$. Da blir selvsagt $I(f)=Q(f)$

Vi sier at kvadraturet har presisjon d om $Q(p)=I(p)$ for alle polynomer av grad $\leq d$

Lagrange-polynomene har presisjon $\geq n$.

Simpsons formel

Intervallet $[-1,1]$ med $t_0=-1,\ t_1=0,\ t_2=1$

$$l_0(t)=\frac{t-t_1}{t_0-t_1}\,\frac{t-t_2}{t_0-t_2}=\frac{t}{-1}\,\frac{t-1}{-2}=\frac{1}{2}t(t-1)$$

$$l_1(t)=\frac{t-t_0}{t_1-t_0}\,\frac{t-t_2}{t_1-t_2}=\frac{t+1}{+1}\,\frac{t-1}{-1}=1-t^2$$

$$l_2(t)=\frac{t-t_0}{t_2-t_0}\,\frac{t-t_1}{t_2-t_1}=\frac{t+1}{2}\,\frac{t}{1}=\frac{1}{2}t(t+1)$$

$w_0=1/3,\ w_1=4/3,\ w_2=1/3$

Dvs:

$$\int_{-1}^{1} f(x)\,dx \approx \frac{1}{3}\left(f(-1)+4f(0)+f(1)\right)$$

Vilkårlig intervall

$$x=\frac{b-a}{2}t+\frac{b+a}{2}$$

$t=-1 \Rightarrow x=a$

$t=1 \Rightarrow x=b$

$$dx=\frac{b-a}{2}\,dt$$

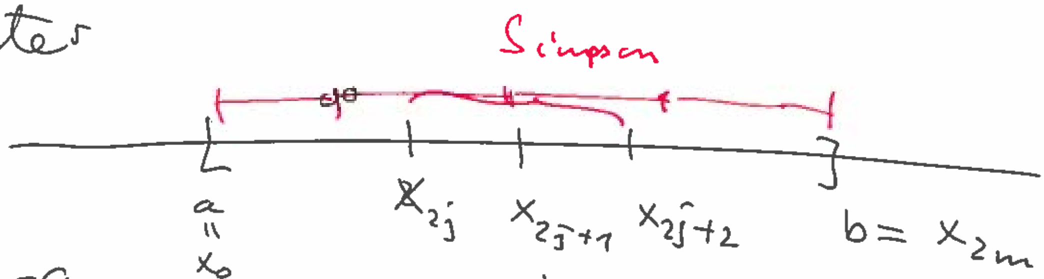
Nå får vi:

$$\int_a^b f(x)\,dx=\frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}t+\frac{b+a}{2}\right)dt$$

$$\approx \frac{b-a}{2}\,\frac{1}{3}\left(f(a)+4f\left(\frac{b+a}{2}\right)+f(b)\right)$$

$$=\frac{b-a}{6}\left(f(a)+4f\left(\frac{b+a}{2}\right)+f(b)\right)$$

Simpson med vilkårlig mange punkter



$$h = \frac{b-a}{2m} \qquad x_j = a + jh \qquad j = 0, \dots, 2m$$

Det betyr:

$$\int_a^b f(x)\,dx = \sum_{j=0}^{m-1} \int_{x_{2j}}^{x_{2j+2}} f(x)\,dx$$

$$\approx \sum_{j=0}^{m-1} \frac{x_{2j+2} - x_{2j}}{6} \left(f(x_{2j}) + 4 f(x_{2j+1}) + f(x_{2j+2}) \right)$$

$$= \sum_{j=0}^{m-1} \frac{2h}{\cancel{6}\,3} \left(f(x_{2j}) + 4 f(x_{2j+1}) + f(x_{2j+2}) \right)$$

$$= \frac{h}{3} \left(f(a) + 2 \sum_{j=1}^{m-1} f(x_{2j}) + 4 \sum_{j=0}^{m-1} f(x_{2j+1}) + f(b) \right)$$