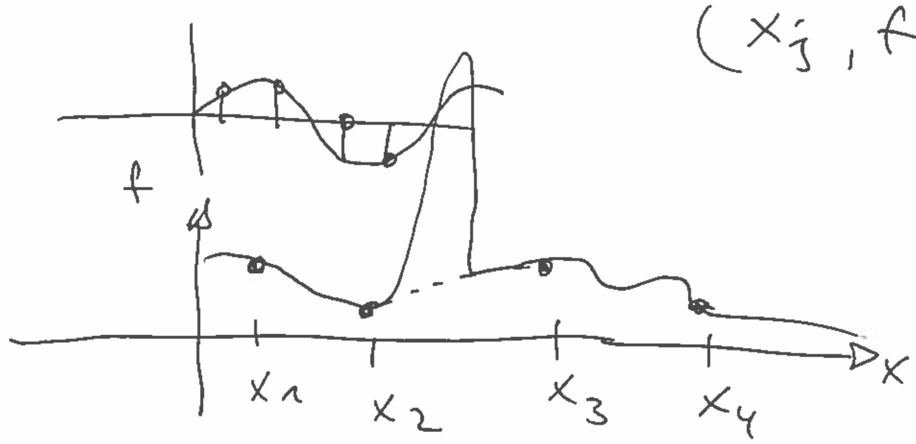
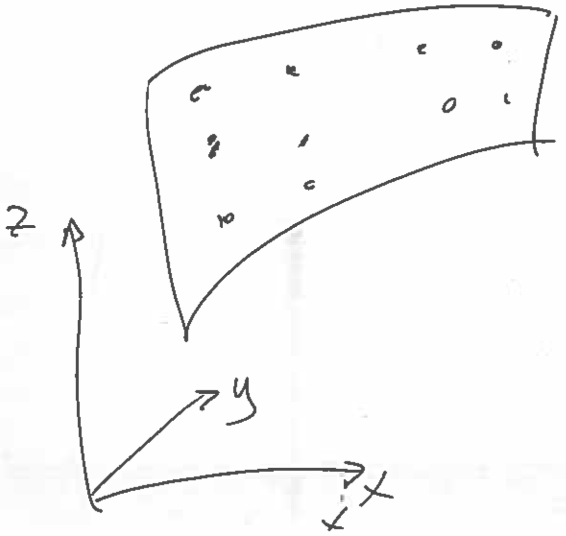
10.10.19

Interpolasjon

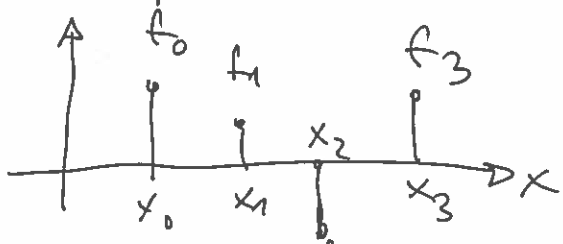
$f(x) = \sin x$

$\left(x_j, \overset{f_j}{\overset{\parallel}{f(x_j)}}\right)$





Gitt punkter (x_j, f_j) $j = 0, \dots, n$



$p_n(x)$

Ønsker å finne et polynom av grad n som går gjennom (x_j, f_j)

$$p_n(x_j) = f_j \quad j = 0, \dots, n$$

Det fins et entydig polynom med denne egenskapen.

Lagrange interpolasjon gir et eksempel på et slikt polynom.

$$p_n(x) = \sum_{j=0}^{n} f_j \underline{\underline{P_j(x)}}$$

der $\quad P_j(x) = 1 \quad$ for $\quad x = x_j$

$\qquad\qquad\quad = 0 \quad$ for $\quad x = x_i, \; i \neq j$

Da blir $\; p_n(x_j) = f_j$

<u>Newton interpolasjon</u>

Vi skal legge til punkter uten å måtte starte all regning på nytt

<u>Induktivt</u>:

Anta at vi har $p_{n-1}(x)$ som er et polynom av grad $n-1$ slik at

$$\underline{p_{n-1}(x_j) = f_j, \quad j = 0, \dots, n-1}$$

Skal konstruere et polynom $p_n(x)$ av grad n slik at $\quad \underline{p_n(x_j) = f_j, \quad j = 0, \dots, n}$

Vi skriver $\quad \underline{p_n(x) = p_{n-1}(x) + g_n(x)}$

$g_n(x) = p_n(x) - p_{n-1}(x)$ er et polynom av grad n

$g_n(x_j) = p_n(x_j) - p_{n-1}(x_j) = f_j - f_j = 0$

for $j = 0, \dots, n-1$

Da kan vi skrive:

$$\underline{g_n(x) = a_n(x - x_0) \dots (x - x_{n-1})}$$

Vi har

$$g_n(x) = p_n(x) - p_{n-1}(x)$$

$$g_n(x_n) = p_n(x_n) - p_{n-1}(x_n) = f_n - p_{n-1}(x_n)$$

$$\parallel$$

$$a_n(x_n - x_0)(x_n - x_1)\dots(x_n - x_{n-1})$$

som gir:

$$a_n = \frac{f_n - p_{n-1}(x_n)}{(x_n - x_0)(x_n - x_1)\dots(x_n - x_{n-1})}$$

Vi har:

$$p_n(x) = p_{n-1}(x) + g_n(x)$$

$$g_n(x) = a_n(x - x_0)\dots(x - x_{n-1})$$

$$a_n = \frac{f_n - p_{n-1}(x_n)}{(x_n - x_0)\dots(x_n - x_{n-1})}$$

der $p_{n-1}(x_j) = f_j, \quad j = 0, \dots, n-1$

$p_n(x_j) = f_j, \quad j = 0, \dots, n$

$n=0$: $\quad p_0(x_0) = f_0 \qquad p_0(x) = f_0$

$n=1$: $\quad p_1(x) = f_0 + a_1(x - x_0)$

$$a_1 = \frac{f_1 - f_0}{x_1 - x_0} = f[x_0, x_1]$$

som gir: $\quad p_1(x) = f_0 + (x - x_0) f[x_0, x_1]$

Sjekk: $\quad p_1(x_1) = f_0 + \cancel{(x_1 - x_0)} \frac{f_1 - f_0}{\cancel{x_1 - x_0}} = f_1$

$n=2$: $\quad p_2(x) = p_1(x) + a_2(x - x_0)(x - x_1)$

$$= f_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) a_2$$

Vi har:

$$a_2 = \frac{f_2 - p_1(x_2)}{(x_2-x_1)(x_2-x_0)}$$

$$= \frac{1}{(x_2-x_1)(x_2-x_0)}\left[f_2 - f_0 - (x_2-x_0)\, f[x_0,x_1]\right]$$

$$= \frac{1}{(x_2-x_1)(x_2-x_0)}\left[f_2 - f_0 - (x_2-x_0)\frac{f_1-f_0}{x_1-x_0}\right]$$

$$= \frac{1}{(x_2-x_0)}\left[\frac{1}{(x_2-x_1)}\left(f_2 - f_0 - \frac{x_2-x_0}{x_1-x_0}(f_1-f_0)\right)\right]$$

$$= \frac{1}{(x_2-x_0)}\left[\frac{f_2}{(x_2-x_1)} - f_0\left(\frac{1}{x_2-x_1} - \frac{x_2-x_0}{(x_1-x_0)(x_2-x_1)}\right) - \frac{x_2-x_0}{(x_2-x_1)(x_1-x_0)} f_1\right]$$

$$= \frac{1}{(x_2-x_0)}\left[\frac{f_2}{(x_2-x_1)} - f_0\,\frac{x_1-\cancel{x_0}-(x_2-\cancel{x_0})}{(x_2-x_1)(x_1-x_0)} - \frac{x_2-x_1+x_1-x_0}{(x_2-x_1)(x_1-x_0)} f_1\right]$$

$$= \frac{1}{(x_2-x_0)}\left[\frac{f_2}{x_2-x_1} + f_0\,\frac{1}{x_1-x_0} - \left(\frac{\cancel{x_2-x_1}}{\cancel{(x_2-x_1)}(x_1-x_0)} + \frac{\cancel{x_1-x_0}}{(x_2-x_1)\cancel{(x_1-x_0)}}\right) f_1\right]$$

$$= \frac{1}{(x_2-x_0)}\left[\frac{f_2}{x_2-x_1} - \frac{f_1}{x_2-x_1} + f_0\,\frac{1}{x_1-x_0} - \frac{f_1}{x_1-x_0}\right]$$

$$= \frac{1}{x_2-x_0}\left(\frac{f_2-f_1}{x_2-x_1} - \frac{f_1-f_0}{x_1-x_0}\right)$$

$$= \frac{1}{x_2-x_0}\left(f[x_1,x_2] - f[x_0,x_1]\right) = f[x_0,x_1,x_2]$$

Vi får da:

$$p_2(x) = f_0 + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1) \cdot f[x_0,x_1,x_2]$$

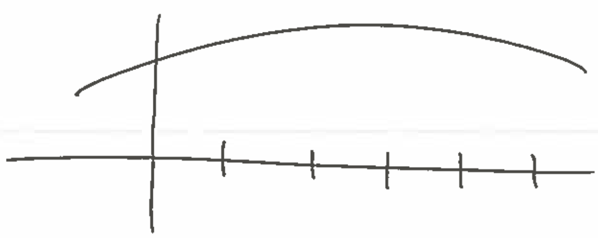
$$p_1(x) = f_0 + (x-x_0)f[x_0,x_1]$$

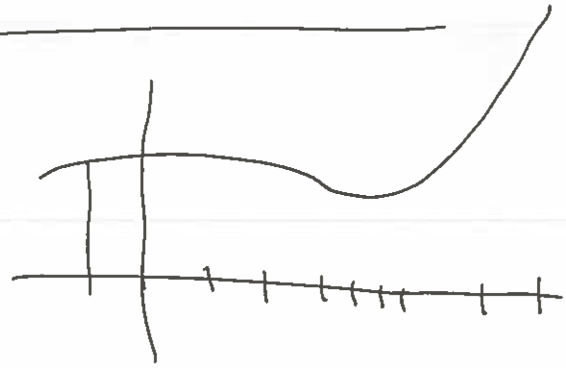
Kan vi nå finne $p_n(x)$?

Man kan vise:

$$p_n(x) = f_0 + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2] + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1})f[x_0,x_1,\dots,x_n]$$

der
$$f[x_0,\dots,x_n] = \frac{f[x_1,\dots,x_n] - f[x_0,\dots,x_{n-1}]}{x_n - x_0}$$





Til nå har x_j vært vilkårlige.

La nå $x_j = x_0 + jh$ (Vi tenker at $0 < h \ll 1$)

Vi definerer: $\Delta f_j = f_{j+1} - f_j$

(forlengs differans)

Vi hadde $\Delta f_j = f_{j+1} - f_j$

Vi kan definere:

$$\underset{\Delta(\Delta f_j)}{\Delta^2 f_j} = \Delta f_{j+1} - \Delta f_j$$

$$= f_{j+2} - f_{j+1} - (f_{j+1} - f_j)$$

$$= f_{j+2} - 2f_{j+1} + f_j$$

Generelt: $\underline{\Delta^k f_j = \Delta^{k-1} f_{j+1} - \Delta^{k-1} f_j}$

Påstand: $\underline{f[x_0, \dots, x_k] = \frac{1}{k!\, h^k} \Delta^k f_0}$

Bevis:

$k=1$: $f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0} = \frac{f_1 - f_0}{h} = \frac{1}{h}(f_1 - f_0)$

$(x_j = x_0 + jh)$

$$\frac{1}{k!\, h^k} \Delta^1 f_0 = \frac{1}{h}(f_1 - f_0)$$

Anta påstanden holder for k:

$$f[x_0, \dots x_k] = \frac{1}{k!\, h^k} \Delta^k f_0$$

Da får vi:

$$f[x_0, \dots, x_{k+1}] = \frac{1}{x_{k+1} - x_0} \left(f[x_1, \dots, x_{k+1}] - f[x_0, \dots, x_k] \right)$$

$$= \frac{1}{(k+1)h} \left(\frac{1}{k!\, h^k} \Delta^k f_1 - \frac{1}{k!\, h^k} \Delta^k f_0 \right)$$

$$= \frac{1}{(k+1)h} \frac{1}{k!\, h^k} \left(\Delta^k f_1 - \Delta^k f_0 \right)$$

$$= \frac{1}{(k+1)!\, h^{k+1}} \Delta^{k+1} f_0$$

Ved induksjon følger påstanden

Det gir: $\quad x = x_0 + rh \quad r \in \mathbb{R}$

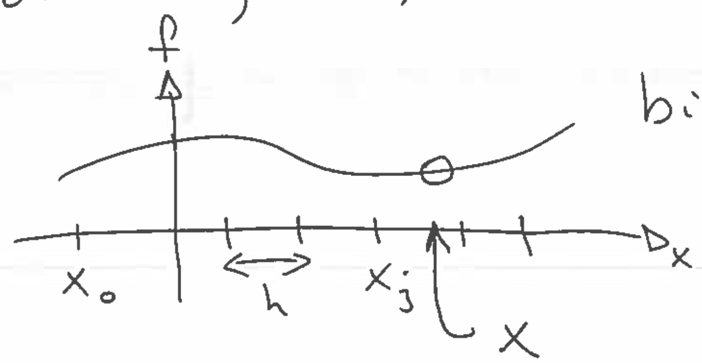
$$\underline{p_n(x)} = f_0 + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2]$$
$$+ \ldots + (x-x_0)\ldots(x-x_{n-1})f[x_0,\ldots,x_n]$$

$$= f_0 + r\cancel{h}\frac{1}{1!\cancel{h}}\Delta f_0 + r\cancel{h}(r-1)\cancel{h}\frac{1}{2!\cancel{h^2}}\Delta^2 f_0$$
$$+ \ldots + r\cancel{h}(r-1)\cancel{h}\ldots(r-(n-1))\cancel{h}\frac{1}{n!\,h^n}\Delta^n f_0$$

$$= f_0 + r\Delta f_0 + \frac{1}{2}r(r-1)\Delta^2 f_0$$
$$+ \ldots + \frac{1}{n!}r(r-1)\ldots(r-(n-1))\Delta^n f_0$$

$$\underline{= \sum_{s=0}^{n}\binom{r}{s}\Delta^s f_0}$$

$$\binom{r}{0} = 1, \quad \binom{r}{s} = \frac{r(r-1)\ldots(r-(s-1))}{s!}, \quad s \in \mathbb{N}$$



binomial-koeffisienter

r stor

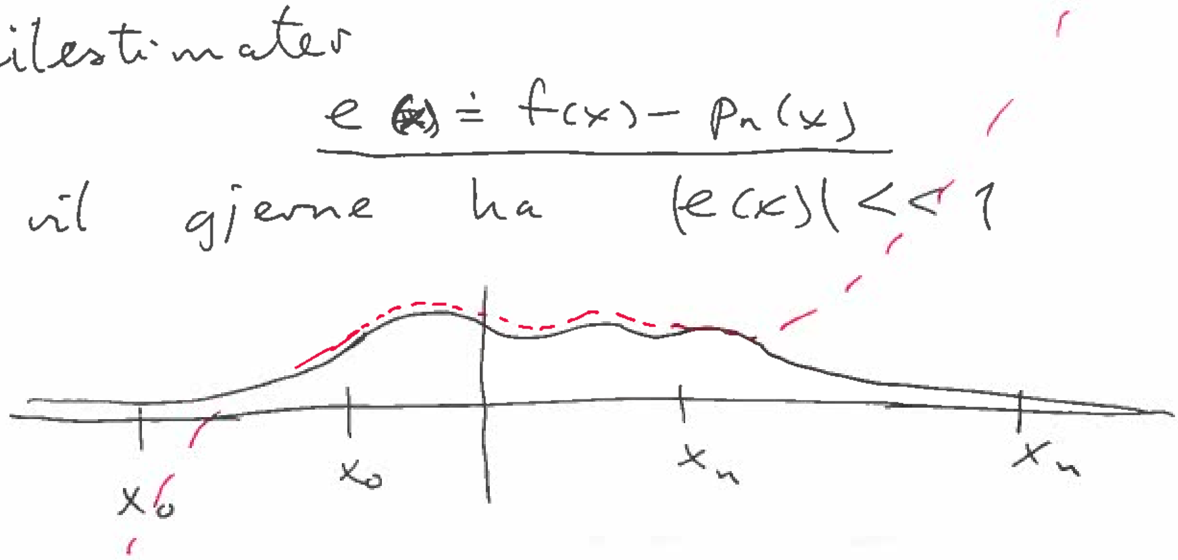
$x = x_0 + rh$

liten

Feilestimater

$$\underline{e(x) \doteq f(x) - p_n(x)}$$

Vi vil gjerne ha $|e(x)| << 1$



Feilestimater:

$$e(x) = f(x) - p_n(x)$$

Vi vet at $e(x_j) = f(x_j) - p_n(x_j)$

$= f_j - f_j = 0, \quad j = 0, \dots, n$

Betrakter funksjonen f på intervallet $[a,b]$, og vi ser på $x \in [a,b]$, men $x \neq x_j$. Definer:

fast pkt!

$$\underline{\varphi(t) = e(t)\,\omega(x) - e(x)\,\omega(t)}$$

der $\omega = (x-x_0)\dots(x-x_n) = \prod_{j=0}^{n}(x-x_j)$

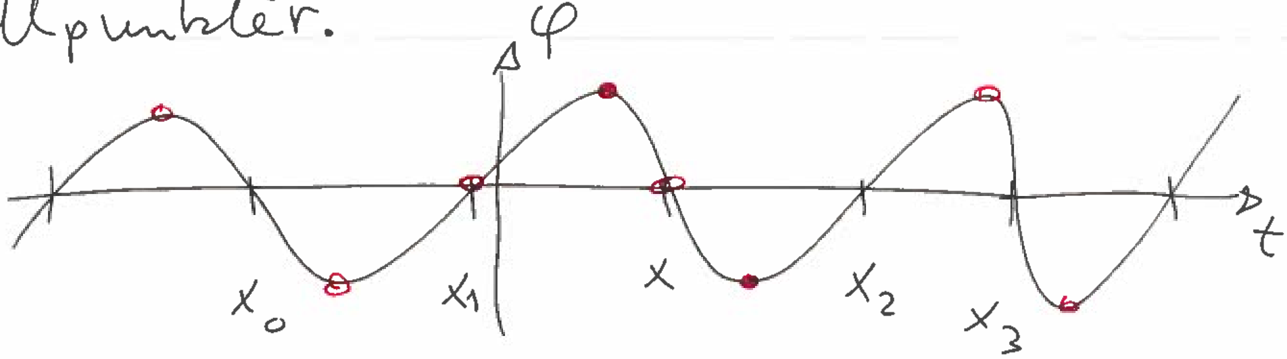
som er polynom av grad $n+1$.

Da blir:

$$\varphi(x_j) = e(x_j)\,\omega(x) - e(x)\,\omega(x_j) = 0, \quad j = 0, \dots, n$$

$$\varphi(x) = e(x)\,\omega(x) - e(x)\,\omega(x) = 0$$

Funksjonen $\varphi(t)$ har minst $n+2$ nullpunkter.



<u>Rolles teorem</u> ~~For en kontinu~~ La φ være en kontinuerlig deriverbar funksjon. Anta at $\varphi(a) = \varphi(b) = 0$. Da fins $z \in (a,b)$ slik at $\varphi'(z) = 0$