Lecture XV

Def (Vector Space)

A Real vector space is a set V with operations

$$+ : V \times V \longrightarrow V$$
$$\cdot : \mathbb{R} \times V \longrightarrow V.$$

They satisfy the conditions : for $\alpha, \beta \in \mathbb{R}$, $z, x, y \in V$

1) $x + y \in V$

2) $x + y = y + x$ 3) $x + (y+z) = (x+y) + z$

4) $\exists\ 0 \in V$: $x + 0 = x$

5) For all $x \in V$ $\exists\ -x \in V$: $(-x) + x = 0$.

6) ~~$\alpha\ V \in V$~~ $\alpha \cdot x \in V$ 7) $\alpha \cdot (\beta x) = (\alpha\beta) \cdot x$

8) $1 x = x$ 9) $\alpha(x + y) = \alpha x + \alpha y$

10) $(\alpha \oplus \beta) x = \alpha x + \beta x$

normal addition in $\mathbb{R}$

addition in V.

Example XV.0.1

$\mathbb{R}^m$, $C[a,b]$, $C(\mathbb{T})$, $\mathbb{P}_n$ = {nth order polynomials over $\mathbb{R}$}.

$\mathbb{P}_n \cong \mathbb{R}^{n+1}$

$\mathbb{P}_0 \subseteq \mathbb{P}_1 \subseteq \mathbb{P}_2 \cdots \subseteq \mathbb{R}(x)$

Def (Norm)

For $x, y \in V$, $\alpha \in \mathbb{R}$, define

$$\|\cdot\| : V \longrightarrow \mathbb{R}_{\geq 0} \quad \text{satisfying}$$

(1) $\|x\| \geq 0$, $\|x\| = 0 \longleftrightarrow x = 0$

(2) $\|\alpha x\| = |\alpha|\ \|x\|$

(3) $\|x + y\| \leq \|x\| + \|y\|$.

Example XVI, 0. 2.

$\|\cdot\|$ on $C([a,b])$

$$d_1(f,g) = \int_a^b |f-g|(x)\,dx$$

$$d_\infty(f,g) = \max_{x\in[a,b]} |f(x)-g(x)|$$

$$d_2(f,g) = \sqrt{\int_a^b |f-g|^2(x)\,dx}$$

Can be effected using norms:

L for Lebesgue

$$\|f\|_{L^1([a,b])} = \int_a^b |f(x)|\,dx$$

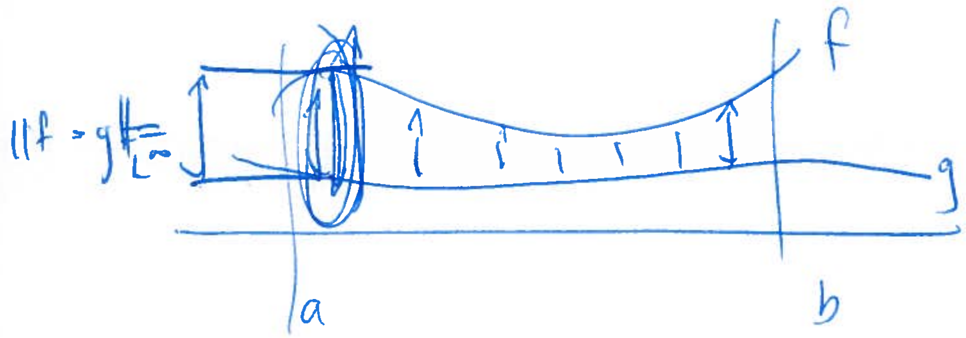
$$\|f\|_{L^2} = \sqrt{\int_a^b |f(x)|^2 dx}$$

$$\|f\|_{L^\infty} = \max_{x\in[a,b]} |f(x)|$$

$$d_1(f,g) = \|f-g\|_{L^1}$$

$$d_2(f,g) = \|f-g\|_{L^2}$$

$$d_\infty(f,g) = \|f-g\|_{L^\infty}.$$

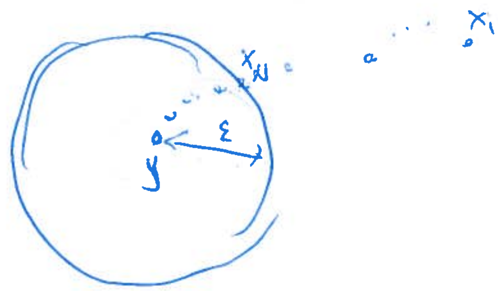


Def (Convergence of a sequence)

Let $\{x_k\}$ be a sequence in $\mathbb{R}$.

$x_k \to y$ ($\{x_k\}$ converges to y)

if $\forall \varepsilon > 0\ \exists N : k > N \to |x_k - y| < \varepsilon$.



Def (Convergence of a Series)

A series $\sum_{n=0}^{\infty} a_n$ converges to y if

$\{b_N\} = \{\sum_{n=0}^{N} a_n\}$ converges to y (as a sequence).

Def (Big-O)

We say that $f(x) = \mathcal{O}(g(x))$ if $\exists M < \infty$:

$$\lim_{x \to \infty} \frac{|f(x)|}{|g(x)|} \leq M$$

or some other specified value

Eg. $f(x) = x$

$g(x) = x \cancel{\log(x)} + 2$

$$\frac{x}{x+2} \to 1$$

We say that $f(x) = o(g(x))$ if

$$\lim_{x \to \infty} \frac{|f(x)|}{|g(x)|} = 0$$

E.g. $f(x) = x$

$g(x) = x \log(x)$

$$\frac{1}{\log(x)} \to 0.$$

$$f(x) = o(g(x)) \underset{\not\Leftarrow}{\Rightarrow} f(x) = \mathcal{O}(g(x))$$

Often $Err(h) = \mathcal{O}(f(h))$

we mean $\lim_{h \to 0}$.

Polynomial Interpolations XV, 1

sampling freq. $= \frac{2\pi}{N}$ ~~sampl~~

and had samples $f(\frac{2\pi}{N}k)$, at $x_k = \frac{2\pi k}{N}$ for $k = 1, \dots, N$.

sought to approximate f by

$$q_{N-1}(x) = \sum_{n=0}^{N-1} c_n e^{inx}$$

requiring $q_{N-1}(x_k) = f(x_k)$.

$$f(x_k) =: f_k = \sum_n M_k^n \, c_n$$

M_k^n is matrix of entries e^{inx_k}

where $k = 1, \dots, N$

$n = 0, \dots, N-1$.

$$c_j = \sum_k (M_k^j)^{-1} f_k$$

EINSTEIN SUMMATION NOTATION — sum over repeated indices implicitly

Eg. $A = (a_{ij}), B = (b_{ij})$. $C = AB$

$$c_{ij} = a_{ik} b_{kj}$$

$\underline{x} = (x_k)$ $\qquad \|x\| = x_k^2 = x_k x_k$

$(= \underline{x} \cdot \underline{x})$

information / sampling points

$$(x_i, y_i) \qquad i = 0, 1, \ldots, N.$$
$$y_i = f(x_i)$$

fit an N^{th} order polynomial

$$q(x) = \sum_{i=0}^{N} c_i x^i$$

i ← not an index (a power!)

c_i: $N+1$ coefficients

for $N+1$ sample points.

Interpolation Problem

Given $n+1$ points, find a polynomial $p(x)$ of lowest possible degree ($p \in \mathbb{P}_n$) satisfying the INTERPOLATION CONDITION

$$p(x_i) = y_i \qquad i = 0, \ldots, n$$

The solution (if it exist) is called the INTERPOLATION POLYNOMIAL and (x_i, y_i) are the INTERPOLATION POINTS.

Example XV.1.

x_i	0	2/3	1
y_i	1	1/2	0

we see that for

$$p_2(x) = \frac{-3x^2 - x + 4}{4}$$

satisfies. $p(x_i) = y_i$

Aside: Direct Approach

$$y_k = M^j_k c_j \qquad (M^j_k) = (x_k)^j$$

for $k = 0, 1, \ldots, N$

XV .2 Lagrange Interpolation

Thm. XV.1

Assume that $n \geq 1$. Let x_i, $i = 0, \ldots, n$ be distinct real numbers, and y_i, $i = 0, \ldots, n$ be real numbers.

Then $\exists !$ polynomial $p_n \in \mathbb{P}_n$ for which

$$p_n(x_i) = y_i \quad \text{for } i = 0, \ldots, N.$$

We prove this via lemma.

Lemma XV.2

Suppose $n \geq 1$. $\exists\ l_k \in \mathbb{P}_n$, $k = 0, \ldots, n$,

such that $l_k(x_i) = \delta_{ik}$ ← KRONECKER DELTA

$$\delta_{ik} = \begin{cases} 1 & i = k \\ 0 & \text{otherwise.} \end{cases}$$

(LEC. $\cancel{\text{XX}}$)

Moreover

$$p_n(x) = \sum_{k=0}^{n} l_k(x)\, y_k$$

is a polynomial satisfying

$$p_n(x_i) = y_i. \quad i = 0, \ldots, n.$$

Proof (of lemma)

For a fixed k, $0 \leq k \leq n$

l_k has zeros at x_i where $i \neq k$.

So
$$l_k(x) = C_k \prod_{\substack{i=0 \\ i \neq k}}^{n} (x - x_i)$$

Since $l_k(x_k) = 1$,

$$C_k = \prod_{\substack{i=0 \\ i \neq k}}^{n} (x_k - x_i)^{-1}$$

So
$$l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

$l_k(x)$ is an nth order polynomial ($\in \mathbb{P}_n$)

$$p_n(x) = \sum_{k=0}^{n} l_k(x)\, y_k \in \mathbb{P}_n$$

($\mathbb{P}_n$ is a vector space)

linear combination of $l_k \in \mathbb{P}_n$.

$$\boxed{p_n(x_j) = \sum_{k=0}^{n} l_k(x_j)\, y_k = \sum_{k=0}^{n} \delta_{kj}\, y_k = 0\cdot y_1 + 0\cdot y_2 \cdots + 1\cdot y_j + 0 \cdots = y_j}$$

$\square$

$l_k(x)$ are called the CARDINAL FUNCTIONS.
or the LAGRANGE INTERPOLATION POLYNOMIALS.

<u>Proof</u> (of theorem)

Existence ✓

Uniqueness: Suppose $q_n \in \mathbb{P}_n$ satisfies

$$q_n(x_k) = y_k.$$

Then $q_n - p_n \in \mathbb{P}_n$

and ~~so~~ $q_n - p_n$ has roots at x_k, $k = 0, \ldots, n$

$\Longleftrightarrow$ $q_n(x_k) - p_n(x_k) = y_k - y_k = 0$

$\hookrightarrow$ $q_n - p_n \in \mathbb{P}_n$ with $n+1$ distinct roots.

$\hookrightarrow$ $q_n - p_n \equiv 0$

Hence $q_n(x) = p_n(x)$. $\square$

So we can speak of <u>the</u> INTERPOLATION POLYNOMIAL.

Example
XV.3

x_i	0	1	3
y_i	3	8	6

$$l_0(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}x^2 - \frac{4}{3}x + 1$$

$$l_1(x) = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}x^2 + \frac{3}{2}x$$

$$l_2(x) = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}x^2 - \frac{1}{6}x.$$

$$p_n(x) = \sum_{k=0}^{n} l_k(x)\, y_k = 3l_0(x) + 8l_1(x) + 6l_2(x)$$

$$= -2x^2 + 7x + 3.$$

XV.3 Newton Interpolation,

(x_0, y_0)

$$f(x) = y_0$$

$$p_0(x_0) = y_0 \leftarrow c_0$$

(x_1, y_1)

$$y_1 = f(x_1) = p(x_1) = \underset{\overset{\shortparallel}{y_0}}{c_0} + c_1(x_1 - x_0)$$

$$c_1 = \frac{y_1 - c_0}{(x_1 - x_0)} = \frac{y_1 - y_0}{x_1 - x_0}.$$

generally

$$p_n(x) = c_0 + \sum_{r=1}^{n} c_r \prod_{j=0}^{r-1} (x - x_j)$$

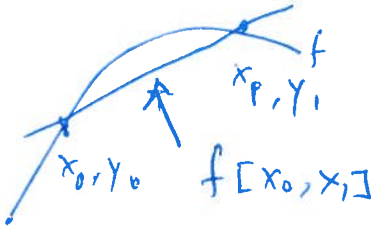
(x_2, y_2)

$$y_2 = p(x_2) = c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1)$$

done! done! what is this?

$$c_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

define
FINITE DIFFERENCES: $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$



$$f[x_0, x_1, \dots, x_n]$$

$$= \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}$$

$$c_k = f[x_0, \dots, x_k]$$