

Lecture XVIII .

Last time:

Newton Cotes formulae

Trapezium Rule

Simpson's Rule

By Lagrange Interpolation

$$P_n(x) = \sum_{k=0}^n l_k(x) f(x_i)$$

$$l_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^n \frac{x - x_j}{x_k - x_j}$$

$$\int_a^b x^n dx = \left. \frac{c}{n+1} x^{n+1} \right|_a^b$$

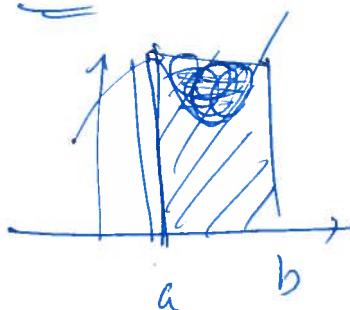
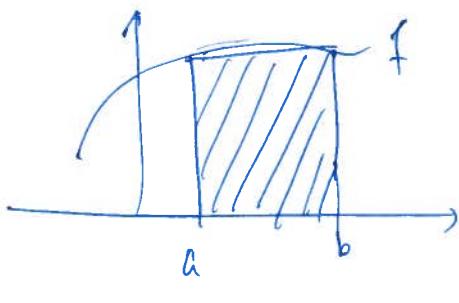
$$\begin{aligned} \int_a^b f(x) dx &\approx \int_a^b P_n(x) dx \\ &= \sum_{i=0}^n \underbrace{\int_a^b l_k(x) dx}_{\text{weights.}} f(x_i) \end{aligned}$$

For $n=1 \rightarrow$ Trapezium rule :

$$x_0 = a, x_1 = b \quad l_0(x) = \frac{x - x_1}{x_0 - x_1} \rightarrow w_0 = \int_a^b l_0(x) dx = \frac{b-a}{2}$$

$$l_1(x) = \frac{x_1 - x}{x_1 - x_0} \rightarrow w_1 = \int_a^b l_1(x) dx = \frac{b-a}{2}$$

$$\int_a^b f(x) dx \approx \frac{b-a}{2} (f(a) + f(b))$$



$Q[f](a, b)$ has degree of precision d if

$$Q[x^k](a, b) = \int_a^b x^k dx \quad \text{for } k \leq d.$$

For $n=2 \rightarrow$ Simpson's Rule:

$$\begin{aligned}x_0 &= a \\x_1 &= \frac{a+b}{2} \\x_2 &= b\end{aligned}$$

$$\begin{aligned}w_0 &= \int_a^b l_0(x) dx \\&= \int_a^b \frac{x-x_1}{x_0-x_1} \frac{x-x_2}{x_0-x_2} dx \\&= \int_{-1}^1 \frac{t(t-1)}{2} \frac{b-a}{2} dt \\&= \frac{b-a}{6} \\w_1 &= \int_a^b \frac{x-x_0}{x_1-x_0} \frac{x-x_2}{x_1-x_2} dx = \frac{4}{6}(b-a)\end{aligned}$$

by symmetry, $w_2 = w_1$,

$$P_n(x) = \sum l_k(x) f(x_i)$$

$$\int_a^b f(x) dx \approx \sum_{i=0}^2 w_i f(x_i) = \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right)$$

Composite Simpson's Rule.

Divided $[a, b]$ into $2m$ subintervals. (of equal length)

$$x_j = a + jh, \quad h = \frac{b-a}{2m}$$

$[x_{2j}, x_{2j+2}]$ $\subset m$ of them

$$\begin{aligned}\int_a^b f(x) dx &= \sum_{j=0}^{m-1} \int_{x_j}^{x_{j+2}} f(x) dx \\&\approx \sum_{j=0}^{m-1} S[f](x_{2j}, x_{2j+2}) \\&= \sum_{j=1}^{m-1} \frac{2h}{6} (f(x_{2j}) + 4f(x_{2j+1}) + f(x_{2j+2})) \\&= \frac{h}{3} (f(x_0) + f(x_{2m}) + 2 \sum_{j=1}^{m-1} f(x_{2j}) \\&\quad + 4 \sum_{j=1}^{m-1} f(x_{2j+1}))\end{aligned}$$

XVIII.2 Error Analysis

we expected

$$\int_a^b f(x) dx \approx \sum_{k=0}^n w_k f(x_k)$$

$$w_k = \int_a^b l_k(x) dx$$

Therefore let us define the error

$$E_n[f](a,b) := \int_a^b f(x) dx - \sum_{k=0}^n w_k f(x_k)$$

Recall that the error bound for the interpolation at

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^n (x - x_i) \omega(x)$$

$\xi = \xi(x) \in [a,b]$

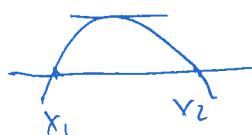
$$\|f - p\|_{L^\infty} = \max_{x \in [a,b]} |f(x) - p(x)|$$

$$\leq \frac{M_{n+1}}{(n+1)!} \| \omega \|_{L^\infty}, \quad M_{n+1} = \| f^{(n+1)} \|_{L^\infty[a,b]}$$

Proof: Consider the auxiliary function

$$\varphi(t) = \frac{f(t) - p_n(t)}{\omega(t)}$$

for $x \neq x_i$.



$$\varphi(t) = 0 \quad \text{for } t = x_i \quad i=0, \dots, n \quad \text{and } t=x$$

$n+2$ points.

Rolle's Theorem

$\varphi'(t)$ has at least $n+1$ zeros

$\varphi^{(n+1)}(t)$ has at least 1 zero. $\longleftrightarrow \xi(x) \in [a,b]$

then $0 = \varphi^{(n+1)}(\xi(x)) = f^{(n+1)}(\xi(x)) - 0 - \frac{f(x) - p_n(x)}{(n+1)!}$

writing $F(t) = f(x(t))$

$$E(a, b) = \frac{b-a}{2} \int_{-t}^t F(s) ds = \frac{1}{3} (F(-t) + 4F(0) + F(t))$$

Set $G(t) = \int_{-t}^t F(s) ds - \frac{t}{3} (F(-t) + 4F(0) + F(t))$

then $E(a, b) = \frac{b-a}{2} G(1)$.

So we need to show that $\exists \xi \in [a, b] :$

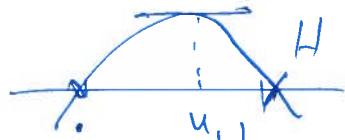
$$\frac{1}{2}(b-a) G(1) = -\frac{f^{(4)}(\xi)}{2880} (b-a)^5.$$

Define $H(t) = G(t) - t^5 f(1)$

Repeatedly apply Rolle's Theorem to H :

$$H(0) = H(1) = 0 \quad \exists u_1 \in (0, 1) \quad H'(u_1) = 0$$

$$H'(u_1) = 0$$



but also $H'(0) = 0$

$$\rightarrow \exists u_2 \in (0, u_1) : H''(u_2) = 0$$

but also $H''(0) = 0$

$$\rightarrow \exists u_3 \in (0, u_2) : H'''(u_3) = 0$$

Now $G^{(3)}(t) = -\frac{1}{3} (F^{(3)}(t) - F^{(3)}(-t))$

so $0 = H'''(u_3) = \frac{-2u_3^2}{3} (F^{(3)}(u_3) - F^{(3)}(-u_3)) - 60 u_3^2 G(1)$

$$w(x) \frac{f^{(n+1)}(\xi(x))}{(n+1)!} = f(x) - p_n(x)$$

$$\max_{x \in [a,b]} |f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \max_{x \in [a,b]} (w(x)).$$

Theorem. Let $f \in C^4([a,b]; \mathbb{R})$

$$\begin{aligned} E[f](a,b) &:= \int_a^b f(x) dx - \left(\frac{b-a}{6} \cdot (f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)) \right) \\ &= \left(-\frac{(b-a)^5}{2880} \right) f^{(4)}(\xi) \end{aligned}$$

for some $\xi \in [a,b]$

$$|E[f](a,b)| \leq \frac{(b-a)^5}{2880} M_4$$

since

$$|f^{(4)}(\xi)| \leq M_4.$$

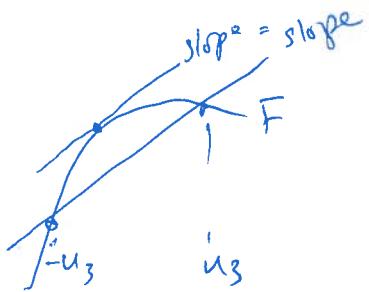
$$M_4 := \|f^{(4)}\|_{L^\infty([a,b])}$$

PF. make again the change-of-variable

$$x = \frac{a+b}{2} + \frac{b-a}{2}t, \quad t \in [-1, 1].$$

so error becomes

$$E(a,b) = \int_a^b f(x) dx - \frac{b-a}{6} (f(a) + 4f\left(\frac{a+b}{2}\right) + f(b))$$



By the mean value theorem ,

$\exists u_4 \in [u_3, u_4] :$

$$0 = H^{(3)}(u_3) = -\frac{2}{3} u_3^2 F^{(4)}(u_4)$$

$$-\ u_3^2 \cdot \frac{2}{3} \cdot 90 = 6(1)$$

$$0 = -\frac{1}{180} F^{(4)}(u_4) - 90 G(1)$$

$$G(1) = -\frac{1}{90} F^{(4)}(u_4)$$

$$= -\frac{(b-a)^4}{1440} f^{(4)}(x(u_4))$$

$$E(a, b) = \frac{(b-a)^5}{2} G(1) = -\frac{(b-a)^5}{2880} f^{(4)}(\xi) \quad \text{. QED } \square$$

Summing up , we get the error for the Composite Simpson's Rule :

$$|E_{\text{comp}}(a, b)| = \left| \sum_{j=0}^{m-1} -\frac{(2\tilde{h})^5}{2880} f^{(4)}(\xi_j) \right| \quad \xi_j \in [x_{2j}, x_{2j+2}]$$

$$\tilde{h} = \frac{b-a}{2m}$$

$$\leq \sum_{j=0}^{m-1} \frac{(2\tilde{h})^5}{2880} M_4$$

$$= \sum_{j=0}^{m-1} 2^5 \frac{(b-a)^5}{2^5 m^5} \frac{1}{2880} M_4$$

$$= m \cdot \frac{(b-a)^5}{m^5} \frac{1}{2880} M_4$$

$$= (b-a) \left(\frac{b-a}{2m} \right)^4 \frac{1}{180} M_4$$

$$R_E = \frac{(b-a)^4}{18n} M_4$$

XVIII.3

Error Estimates

of the error we have 2 concerns:

- ① bound on $E(a,b)$ is usually too big
- ② Not always quick to find $|f^{(4)}(x)|$

Let $S_m(a,b)$ be the composite
Simpson's Rule on $2m$ subintervals
(i.e. Rule is applied m times)

$$E_1(a,b) := I(a,b) - S_1(a,b) \approx C(b-a)^5 =: E_1(a,b)$$

$$E_2(a,b) := I(a,b) - S_2(a,b) \approx 2C\left(\frac{b-a}{2}\right)^5 =: E_2(a,b)$$

$$\frac{S_2(a,b) - S_1(a,b)}{S_2(a,b) - S_1(a,b)} = \frac{1}{16} E_1(a,b)$$

$$\approx \frac{15}{16} C(b-a)^5$$

$$C(b-a)^5 \approx \frac{16}{15} (S_2 - S_1)$$

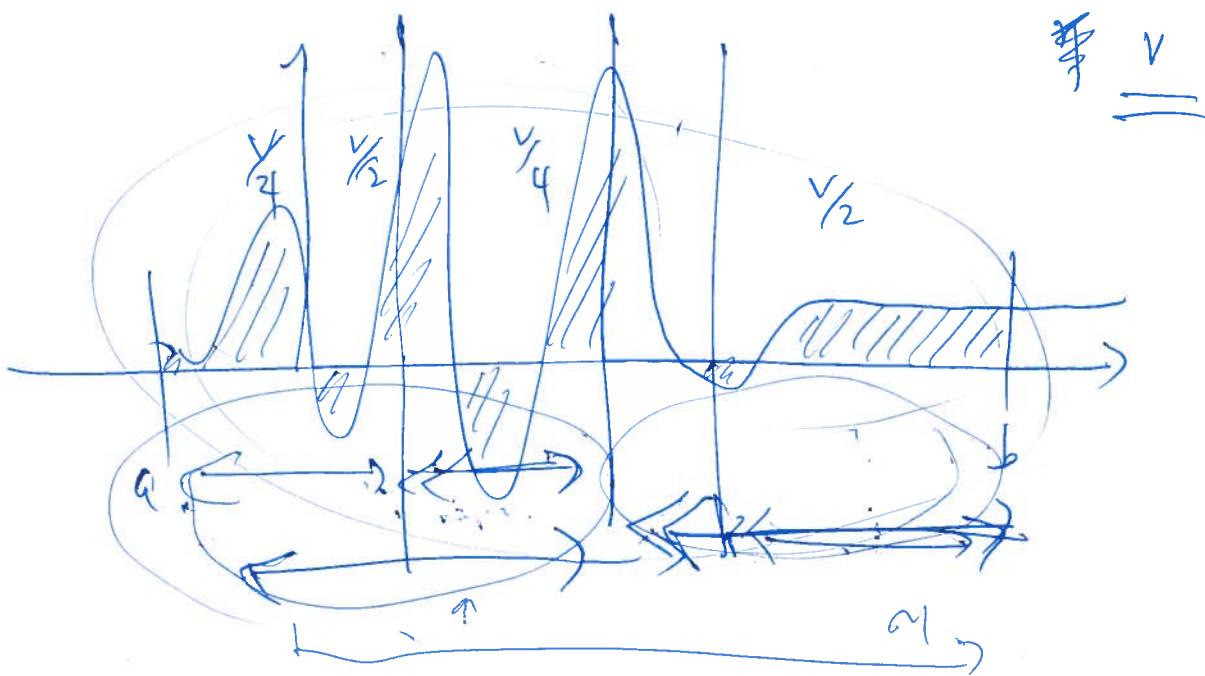
$$\text{so } E_1(a,b) \approx \frac{16}{15} (S_2 - S_1)$$

$$E_2(a,b) \approx \frac{1}{15} (S_2 - S_1)$$

$$\frac{1}{15} (S_2 - S_1) \approx E_2(a,b) = I(a,b) - S_2(a,b)$$

$$\text{so } I(a,b) \approx \frac{16}{15} S_2 - \frac{1}{15} S_1$$

$$= S_2 + \frac{1}{15} (S_2 - S_1)$$



pre-set an error tolerance T_{tol}

partition $[a, b]$ into intervals of length
adapted to the local behaviour of f .

$$a_0 = X_0 < X_1 < \dots < X_m = b.$$

$$\underline{E(a,b)} \leq \sum |\varepsilon(X_j, X_{j+1})| = \sum \frac{|X_{k+1} - X_k|}{b-a} \cdot \Phi_0 |$$

$$\Phi_0 = T_{\text{tol}}$$

then we expect

$$E(a,b) \approx \sum_{k=0}^{m-1} \varepsilon(X_k, X_{k+1})$$

$$\leq \sum_{k=0}^{m-1} |\varepsilon(X_k, X_{k+1})|$$

$$= T_{\text{tol}}.$$

def Approx (a, b, Tol)

Calculate S(a,b) and E(a,b)

If $|E(a,b)| \leq Tol$, stop.

return

$$I(a,b) = S(a,b) + E(a,b)$$

else

$$I(a,b) = \text{Approx} \left(a, \frac{a+b}{2}, \frac{Tol}{2} \right)$$

$$+ \text{Approx} \left(\frac{a+b}{2}, b, \frac{Tol}{2} \right)$$

XVIII. Non Newton Cotes Quadratures.

Hermite Interpolation

$$\{x_i, f(x_i), f'(x_i)\}_i^n$$

$$P_{2n+1} \leftarrow P_{2n+1}$$

Gauss-Legendre quadrature.

sample at roots of Legendre polynomials

} recursively
defining adapt
the partition