

Lecture XII

Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(wave speed)²



$$u(t_0) = u(t, L) = 0$$

$$u(0, x) = f(x)$$

$$\partial_t u(0, x) = g(x)$$

Questions

i) Existence

ii) Uniqueness

iii) Continuous Dependence

iv) Asymptotics

$$u(t, x) = \sum_{n \geq 0} \frac{c_n}{2} \left(\sin \left(\frac{n\pi}{L} (x+ct) \right) + \sin \left(\frac{n\pi}{L} (x-ct) \right) \right) \\ + \frac{D_n}{2} \left[\cos \left(\frac{n\pi}{L} (x+ct) \right) - \cos \left(\frac{n\pi}{L} (x-ct) \right) \right]$$

$$= \frac{1}{2} (f_-(x+ct) + f_-(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(r) dr$$

$$u(t, x) = F(x) G(t)$$

Convention:

$$\square = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \quad \text{D'ALEMBERTIAN}$$

$$\text{wave eq.: } \square u = 0$$

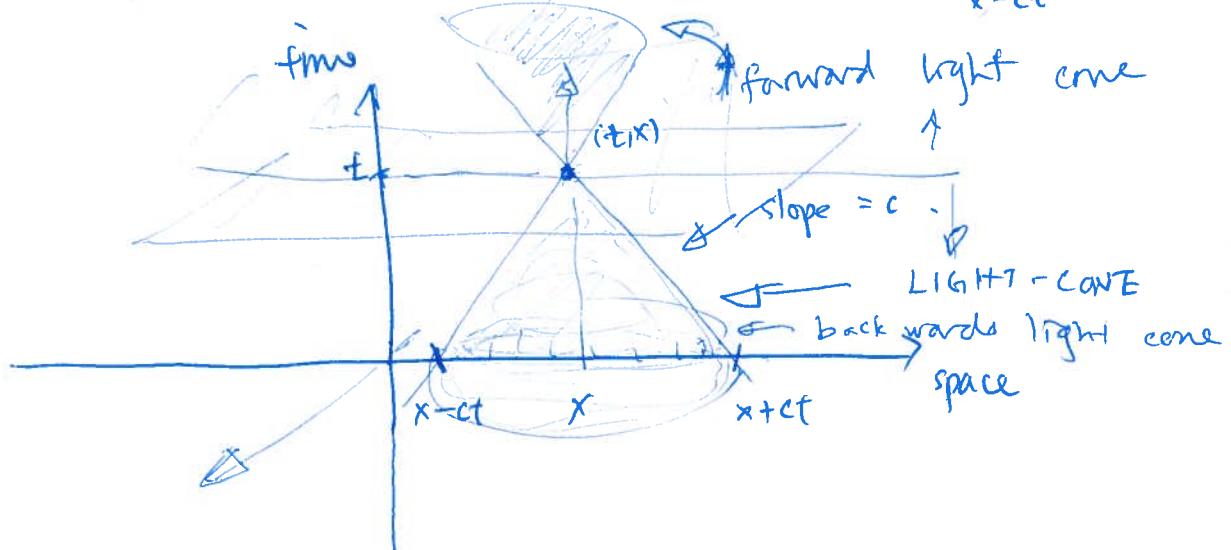
1) no dispersion for wave equation.

2) finite speed of propagation

$u(t, x)$ only depends on information coming from within the triangle formed by the points

↗ $(0, x-ct), (0, x+ct), (t, x)$.
time space

$$u(t, x) = \frac{1}{2} (f_-(x+ct) + f_-(x-ct)) + \frac{1}{2} c \int_{x-ct}^{\infty} g(r) dr$$



In higher ($d \geq 3$) dimensions.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \sum_{n=1}^d \frac{\partial^2 u}{\partial x_n^2}$$

in odd dimensions ≥ 3 — only surface of light cone
in even dimensions (≥ 2) — entire backwards light cone.

Huygen's Principle.

XII.1 METHOD OF CHARACTERISTICS

change coordinates \rightarrow make equation easier to solve

★
change back once we
have solution

mini example:

algebraic equation $\rightarrow x^2 + bx + c = 0$

~~x~~ completing the squares : $x - \frac{b}{2} \mapsto y$

$$x^2 + bx + b^2/4 + (c - b^2/4) = (x - b/2)^2 + (c - b^2/4)$$

$$= y^2 + (c - b^2/4)$$

$$y^2 = b^2/4 - c$$

$$y = \pm \sqrt{b^2/4 - c}$$

$$\rightarrow x = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$$

Example XII .1

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$y = x + ct, \quad z = x - ct$$

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t} \right) \\ &= \frac{\partial}{\partial t} \left(c \frac{\partial u}{\partial y} + (-c) \frac{\partial u}{\partial z} \right) \\ &= \frac{\partial}{\partial y} \left(c \frac{\partial u}{\partial y} + (-c) \frac{\partial u}{\partial z} \right) \frac{\partial y}{\partial t} + \frac{\partial}{\partial z} \left(c \frac{\partial u}{\partial y} + (-c) \frac{\partial u}{\partial z} \right) \frac{\partial z}{\partial t} \\ &= c^2 \frac{\partial^2 u}{\partial y^2} - c^2 \frac{\partial^2 u}{\partial y \partial z} + c^2 \frac{\partial^2 u}{\partial z \partial y} + c^2 \frac{\partial^2 u}{\partial z^2} \\ &= c^2 \left(\frac{\partial^2 u}{\partial y^2} - 2 \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z^2} \right).\end{aligned}$$

similarly,

$$\frac{\partial^2 u}{\partial x^2} = \cancel{c^2} \left(\frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z^2} \right).$$

$$0 = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \cancel{-4c^2} \frac{\partial^2 u}{\partial y \partial z}$$

wave eq.
in y-z
coordinates

$$0 = \frac{\partial^2 u}{\partial y \partial z}$$

$$\frac{\partial}{\partial z} \left(\frac{\partial u}{\partial y} \right) = 0$$

$$\frac{\partial u}{\partial y} = h(y)$$

will find
out using
bound/ init.
conditions

$$\frac{\partial u}{\partial y} = h(y)$$

$$u = \underbrace{\int h(y) dy}_{\Phi(y)} + \Psi(z)$$

$$\Phi(y)$$

$$u(t, x) = \Phi(y) + \Psi(z) = \underbrace{\Phi(x+ct)}_{\Phi} + \Psi(x-ct)$$

D'ALEMBERT'S SOLUTION

Remarks to apply initial conditions to determine
 Φ , Ψ

(just like we used these conditions to determine
 $\{C_n\}$, $\{D_n\}$)

$$\begin{aligned} u(0, x) &= f(x) \Rightarrow \boxed{\Phi(x) + \Psi(x) = f(x)} \\ \partial_t u(0, x) &= g(x) \Rightarrow c \Phi'(x) - c \Psi'(x) = \boxed{g(x)} \\ \Phi'(x) - \Psi'(x) &= \frac{1}{c} g(x) \end{aligned}$$

\downarrow integrate

$$\boxed{\Phi(x) = \Psi(x)}$$

$$\begin{aligned} \Phi(x) - \Psi(x) &= \Phi(x_0) - \Psi(x_0) + \frac{1}{c} \int_{x_0}^x g(r) dr \\ \Phi(x) + \Psi(x) &= f(x) \end{aligned}$$

SOLVE \Downarrow LINEAR SYSTEM!

$$\begin{aligned} \Phi(x) &= \frac{1}{2} (f(x) + \Phi(x_0) - \Psi(x_0)) + \frac{1}{2c} \int_{x_0}^x g(r) dr \\ \Psi(x) &= \frac{1}{2} (f(x) - (\Phi(x_0) - \Psi(x_0))) - \frac{1}{2c} \int_{x_0}^x g(r) dr \end{aligned}$$

$$u(t, x) = \Phi(x+ct) + \Psi(x-ct)$$

$$= \frac{1}{2} (f(x+ct) + f(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(r) dr$$

if $g = 0 \rightarrow$ boundary conditions
 enforces f has to be odd

UNIQUENESS : if u_1, u_2 are two solutions with
 the same initial & boundary data,
 then

$$(u_1 - u_2)(0, x) = f(x) - f(x) = 0$$

$$\partial_t(u_1 - u_2)(0, x) = g(x) - g(x) = 0$$

$w = u_1 - u_2$ is also a solution as
 wave eq is linear

D'Alembert's $\Rightarrow w = 0 \Rightarrow u_1 = u_2 \rightarrow$ solution is unique.

XII .2 Method of Characteristics for quasilinear second order equations in two variables

$$A(x,y) \frac{\partial^2 u}{\partial x^2} + 2B(x,y) \frac{\partial^2 u}{\partial x \partial y} + C(x,y) \frac{\partial^2 u}{\partial y^2} = F(x,y,u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$$

i) change of coordinates

↓

ii) equation becomes easy to integrate

↓

iii) solve and change back

change of coordinates

$$\xi = \varphi(x,y), \eta = \psi(x,y)$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} \right) \\ &= \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} \right) \frac{\partial \xi}{\partial x} \\ &\quad + \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} \right) \frac{\partial \eta}{\partial x} \\ &= \left(\frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta \partial \xi} \frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \eta}{\partial x} \right) \frac{\partial \xi}{\partial x} \\ &\quad + \left(\frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial^2 \xi}{\partial x \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \eta}{\partial x} \right) \frac{\partial \eta}{\partial x} \\ &= \underbrace{\frac{\partial^2 u}{\partial \xi^2}}_{= M} (\frac{\partial \xi}{\partial x})^2 + \underbrace{2 \frac{\partial^2 u}{\partial \eta \partial \xi}}_{= M} (\frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial x}) + \underbrace{\frac{\partial^2 u}{\partial \eta^2}}_{= M} (\frac{\partial \eta}{\partial x})^2 \end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2} = \begin{pmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \eta}{\partial x} \end{pmatrix}^T \underbrace{\begin{pmatrix} \frac{\partial^2 u}{\partial \xi^2} & \frac{\partial^2 u}{\partial \xi \partial \eta} \\ \frac{\partial^2 u}{\partial \eta \partial \xi} & \frac{\partial^2 u}{\partial \eta^2} \end{pmatrix}}_M \begin{pmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \eta}{\partial x} \end{pmatrix}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial y} &= \begin{pmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \eta}{\partial x} \end{pmatrix}^T M \begin{pmatrix} \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial y} \end{pmatrix} \\ \frac{\partial^2 u}{\partial y^2} &= \begin{pmatrix} \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial y} \end{pmatrix}^T M \begin{pmatrix} \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial y} \end{pmatrix} \end{aligned}$$

Jacobian matrix

grouping the terms,

$$\underbrace{a(\xi, \eta)}_{A} \frac{\partial^2 u}{\partial \xi^2} + \underbrace{b(\xi, \eta)}_{B} \frac{\partial^2 u}{\partial \xi \partial \eta} + \underbrace{c(\xi, \eta)}_{C} \frac{\partial^2 u}{\partial \eta^2} = \tilde{F}(\xi, \eta, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta})$$

choose $\xi = \varphi(x, y)$ $\eta = \psi(x, y)$ such that

$$a(\xi, \eta) = c(\xi, \eta) = 0$$

$$D_\xi = \frac{\partial \xi / \partial x}{\partial \xi / \partial y}, \quad D_\eta = \frac{\partial \eta / \partial x}{\partial \eta / \partial y}$$

$$a(\xi, \eta) = \left(\frac{\partial \xi}{\partial y} \right)^2 (A(x, y) D_\xi^2 + 2B(x, y) D_\xi D_\eta + C(x, y))$$

$$b(\xi, \eta) = \frac{\partial \eta}{\partial y} \frac{\partial \xi}{\partial y} (A(x, y) D_\xi D_\eta + B(x, y) (D_\eta^2 + D_\xi^2) + C(x, y))$$

$$c(\xi, \eta) = \left(\frac{\partial \eta}{\partial y} \right)^2 (A(x, y) D_\eta^2 + 2B(x, y) D_\eta + C(x, y))$$

gives us
"good"
change of
coordinates

$$\begin{cases} A D_\xi^2 + 2B D_\xi + C = 0 \\ A D_\eta^2 + 2B D_\eta + C = 0 \end{cases}$$

$$AD^2 + 2BD + C = 0 \rightarrow D_{\pm} = \frac{-B \pm \sqrt{B^2 - AC}}{A}$$

$$D_\xi = D_+, \quad D_\eta = D_-$$

$$\text{Notice : } b^2 - ac = (B^2 - AC) \det(M)^2$$

sign $(b^2 - ac)$ is preserved by change of coordinate

$$\begin{cases} B^2 - AC > 0 & - \text{real roots} - \text{Eq. is HYPERBOLIC} \\ B^2 - AC < 0 & - \text{complex roots} - \text{Eq. is ELLIPTIC} \\ B^2 - AC = 0 & - \cancel{\text{mult.}} \text{ roots} - \text{Eq. is PARABOLIC} \end{cases}$$

So along the curves $\xi = \varphi(x, y)$, $\eta = \psi(x, y)$

where ξ, η are kept constant,
the transformed PDE is

$$a(\xi, \eta) \frac{\partial^2 u}{\partial \xi^2} + 2b(\xi, \eta) \frac{\partial^2 u}{\partial \xi \partial \eta} + c(\xi, \eta) \frac{\partial^2 u}{\partial \eta^2} = \tilde{F}(\xi, \eta, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta})$$

wave equation: $\xi = x + cy$

$$\eta = x - cy$$

$$\frac{\partial \xi / \partial x}{\partial \xi / \partial y} = D_\xi = \frac{-B + \sqrt{B^2 - AC}}{A} \quad D_\eta = \frac{-B - \sqrt{B^2 - AC}}{A}$$

$$\frac{\partial \eta / \partial x}{\partial \eta / \partial y}$$

along the curves: $0 = d\xi = \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy \rightarrow \frac{dy}{dx} = -D_\xi$
 $0 = d\eta = \frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial y} dy \rightarrow \frac{dy}{dx} = -D_\eta$

$$A D_\xi^2 + 2BD_\xi + C = 0$$

$$A D_\eta^2 + 2BD_\eta + C = 0$$

equation for good transformation:

$$A \left(\frac{dy}{dx} \right)^2 + 2B \frac{dy}{dx} + C = 0$$

CHARACTERISTIC EQUATION

curves are called characteristics.

transformed equations — NORMAL / CANONICAL / STANDARD FORM