Last Time

Some properties of the Fourier Transform on $\mathbb{R}$:

i) Differentiation $\rightsquigarrow$ multiplicative factor

$$\widehat{\frac{df}{dx}}(\omega) = i\omega \hat{f}(\omega)$$

$$\widehat{(x \cdot f)} \to \widehat{\mathrm{Id} \cdot f}(\omega) = i \frac{d}{d\omega} \hat{f}(\omega)$$

$\mathrm{Id}(x) = x$

ii) Convolution $\rightsquigarrow$ multiplication

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)\, g(y)\, dy$$

$$\widehat{f * g}(\omega) = \sqrt{2\pi}\, \hat{f}(\omega)\, \hat{g}(\omega)$$

$$\widehat{fg}(\omega) = \frac{1}{\sqrt{2\pi}} (\hat{f} * \hat{g})(\omega)$$

iii) ~~FT~~ Translation (shift) $\rightsquigarrow$ exponential factor

$$\widehat{f(\cdot + a)}(\omega) = e^{ia\omega} \hat{f}(\omega)$$

$$\widehat{e^{ia\cdot} f(\cdot)}(\omega) = \hat{f}(\omega - a)$$

iv) Parseval's Identity

$$\int_{\mathbb{R}} |f|^2 dx = \int_{\mathbb{R}} |\hat{f}|^2 d\omega.$$

Discrete Fourier Transform

$\underline{f} = (f_0, f_1, \ldots, f_{N-1})$ $\qquad x_k = \frac{2\pi k}{N}$ sampling points

~~$\hat{\underline{f}} = \underline{\underline{F_N}} \cdot \underline{f}$~~

$$\hat{\underline{f}} = \underline{\underline{F_N}}\, \underline{f}$$

matrix

$$\hat{f}_m = \sum_{k=0}^{N-1} f_k\, e^{-2\pi i m k/N}$$

Inverse : $\underline{\underline{F_N}}^{-1} = \frac{1}{N} \overline{F_N}$ ← complex conjugate

Turning a $O(N^2)$ problem $\longrightarrow O(N \log(N))$

Divide & conquer :

Stategy involves
splitting the problem into N pieces at lowest level
takes N steps

↓

$\log_2(N)$ levels $\left\{ \begin{array}{l} \text{recombine by pairs} \\ \text{solve } N/2 \text{ problems} \rightarrow \text{take } N \text{ steps} \end{array} \right.$

→ total $N \log_2(N)$

"Fast Fourier Transform"

Lecture XI

WAVE EQUATION:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

evolution equations

$$u_0 \longrightarrow u(t,x)$$

Most PDEs cannot be solved analytically / explicitly

Yes then "well-posed"

i) Existence: given an (a class of) initial condition(s)/ boundary data, is there a solution (up to some finite time T)?

ii) Uniqueness: given ~~an~~ initial data, if a solutions exists, is it unique?

iii) Continuous dependence: if an initial condition changes a little (as measured by some distance), does the solutions change just a little?

iv) Asymptotic Behavior: is there a steady state? chaotic behavior? asymptotic "blow-up" $\nearrow \infty$?

v) Symmetries, Special Solutions / Phenomena:

Are there self-similar behavior

Solitary wave (solitons)

Conserved quantities

multiple-scales

PDEs can be thought of as infinite-dimensional versions of ODEs

WAVE EQUATION:

$$\partial_t^2 u = c^2 \partial_x^2 u$$

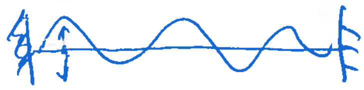
models small deviations $u(t,x)$ from a straight line of a mildly elastic cord under the effects of tension

Assumptions

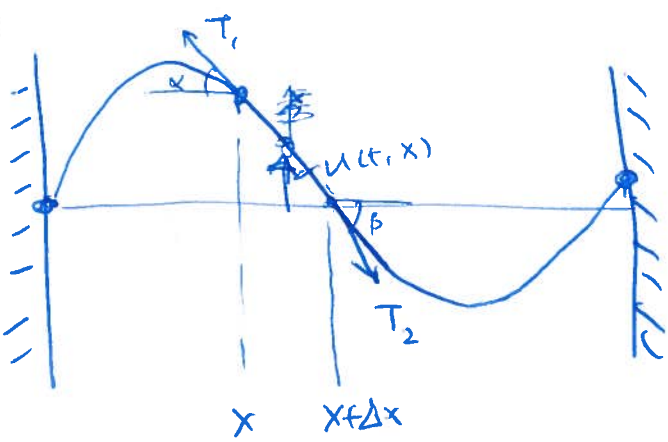
(i) The cord is a perfect cord — does not resist bending

- is uniform linear density λ (unit mass / unit length)
- zero cross section
- fixed at both ends

(ii) each infinitesimal segment of the cord only moves transversely in the plane.



(iii) the cord is perfectly straight at rest, of rest length L.



$$T_1 \cos(\alpha) = T_2 \cos(\beta) =: T$$

Net vertical force ~~/acceleration~~ = acceleration × mass

Newton's 2nd law $F = ma$

$$m = \lambda \cdot \Delta x \,, \quad \text{acceleration} = \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\lambda \cdot \Delta x}{T} \frac{\partial^2 u}{\partial t^2} = \frac{(T_2 \sin(\beta) - T_1 \sin(\alpha))}{T}$$

$$= \tan(\beta) - \tan(\alpha).$$

$$\tan(\alpha) = \frac{\partial u}{\partial x}\Big|_x \qquad \tan(\beta) = \frac{\partial u}{\partial x}\Big|_{x+\Delta x}$$

$$\frac{\lambda}{T} \frac{\partial^2 u}{\partial t^2} = \frac{1}{\Delta x}\left(\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_x \right)$$

$$\longrightarrow \frac{\partial^2 u}{\partial x^2}$$

Constant λ/T is non-negative by construction.

denote ~~it~~ it by $1/c^2$

$$[T] = \text{mass} \times \frac{\text{distance}}{(\text{time})^2}$$

$$[\lambda] = \frac{\text{mass}}{\text{distance}}$$

$$\boxed{\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}}$$

$$[c^2] = \underline{(\text{speed})^2}$$

c – wave speed

ii) $\begin{cases} u(t,0) = 0 \,, \quad u(t,L) = 0 & \text{— boundary conditions} \\ u(0,x) = f(x) \quad (\partial_t u)(0,x) = g(x) & \text{— initial conditions} \end{cases}$

f & g are continuous, defined over $[0,L]$

CAUCHY PROBLEM for the wave equation.

XI.2 Solving the Equation.

separation of variables

ansatz: $u(t,x) = F(x)\, G(t)$

insert $F(x)\, G(t)$ into the equation:

$$F \frac{d^2 G}{dt^2} = c^2 \frac{d^2 F}{dx^2} G$$

we must conclude

$$\underbrace{\frac{1}{G}\frac{d^2 G}{dt^2}}_{\text{purely a function of } t} = \underbrace{c^2 \frac{1}{F}\frac{d^2 F}{dx^2}}_{\text{purely a function of } x} = k$$

$$0 = \frac{d^2 G}{dt^2} - k G \qquad 0 = \frac{d^2 F}{dx^2} - \frac{k}{c^2} F$$

---

consider boundary conditions:

$u(t,0) = 0$, $u(t,L) = 0$ – fixed at both ends.

either $F(0), F(L) = 0$ or $G(t) = 0$

$\Downarrow$ trivial solutions

if $k > 0$

$F(x) = A e^{-\sqrt{k}x} + B e^{\sqrt{k}x}$ → also trivial solutions

→ $k \leq 0$ k is an eigenvalue

$$\frac{d^2 F}{dx^2} = \frac{k}{c^2} F$$

box: $[0, L]$ → forces $k = -p^2$ to take discrete values.

$k \leq 0$ → $F(x) = A \cos(px) + B \sin(px)$

$F(0) = 0$ → $A = 0$ $F(L) = 0$ → $\left\{ p = \frac{n\pi}{L} \right\}$

SPECTRUM
NORMAL MODES.

$$u(t,x) = F(x)\,G(t)$$

$$0 = \frac{d^2 G}{dt^2} - c^2 k\, G \qquad 0 = \frac{d^2 F}{dx^2} - k F$$

$\Downarrow$ using $k = -p^2 = -\frac{n^2\pi^2}{L^2}$

$\Downarrow$

$$G_n(t) = C_n^* \cos\left(\frac{cn\pi}{L} t\right) + D_n^* \sin\left(\frac{cn\pi}{L} t\right)$$

$$u(t,x) = \sum_{n=0}^{\infty} F_n(x) \cdot G_n(t)$$

$$= \sum_{n \geq 0} \left[C_n \cos\left(\frac{cn\pi}{L} t\right) + D_n \sin\left(\frac{cn\pi}{L} t\right)\right] \sin\left(\frac{n\pi}{L} x\right)$$

$$C_n = B_n C_n^* \quad , \quad D_n = B_n D_n^*$$

it remains to match initial & boundary conditions to find $\{C_n, D_n\}$.

Condition $u(0,x) = f(x)$, on $[0,L]$,

$\longrightarrow$ $f(x) = \sum_{n>0} C_n \sin\left(\frac{n\pi}{L} x\right)$ $\Leftarrow$ SINE SERIES!

$\longrightarrow$ $C_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$

ODD EXTENSION: $f_-(x) = \sum_{n>0} C_n \sin\left(\frac{n\pi}{L} x\right) = f(x)$ on $[0,L]$

$\partial_t u(0,x) = g(x)$ on $[0,L]$:

$$\partial_t u(t,x) = \sum_{n>0} \frac{cn\pi}{L}\left(-C_n \sin\left(\frac{cn\pi}{L} t\right) + D_n \cos\left(\frac{cn\pi}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right)$$

Set $t=0$

$g(x) = \sum_{n>0} \frac{cn\pi}{L} D_n \sin\left(\frac{n\pi}{L} x\right)$ $\Leftarrow$ SINE SERIES! AGAIN!

$$c\frac{n\pi}{L} D_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$D_n = \frac{2}{cn\pi}\int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$u(t,x) = \sum_{n\geq 0} \left(C_n \cos\left(\frac{cn\pi}{L} t\right) + D_n \sin\left(\frac{cn\pi}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right)$$

Trigonometric identities:

$$u(t,x) = \sum_{n>0} \frac{C_n}{2}\left(\sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\left(\frac{n\pi}{L}(x-ct)\right)\right)$$
$$+ \frac{D_n}{2}\left(\cos\left(\frac{n\pi}{L}(x-ct)\right) - \cos\left(\frac{n\pi}{L}(x+ct)\right)\right)$$

Observe:

$$\cancel{\sum_{n>0} \left(C_n \cos\left(\frac{cn\pi}{L} t\right) + D_n \sin\left(\frac{cn\pi}{L}\right.\right.}$$

$$\sum_{n>0} \frac{C_n}{2}\left(\sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\left(\frac{n\pi}{L}(x-ct)\right)\right)$$
$$= \frac{1}{2}\left(f_-(x+ct) + f_-(x-ct)\right)$$

if $x+ct$ is constant $\qquad$ $x-ct$ is constant

$$\sum_{n>0} \frac{D_n}{2}\left(\cos\left(\frac{n\pi}{L}(x-ct)\right) - \cos\left(\frac{n\pi}{L}(x+ct)\right)\right)$$
$$= -\frac{1}{2c}\int_{x+ct}^{x-ct} g(r)\,dr$$

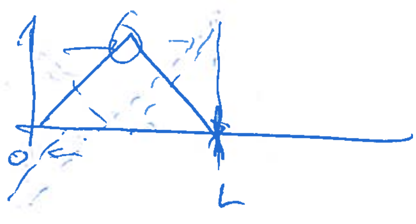
$$u(t,x) = \frac{1}{2}\left(f_-(x+ct) - f_-(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(r)\,dr$$

if f ~~twice~~ is twice continuously differentiable

if g is once continuously differentiable

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

"generalized solutions"



(i) Existence ✓

(ii) Uniqueness ?

(iii) if f & g changes a little in, say, does:

f^{ε}, g^{ε} $\sup_{[0,L]} |f^{\varepsilon} - f| < \varepsilon$ → solution also changes little

Continuous dependence ✓

(iv) asymptotics ✓