Lecture XXIV

This time: solving boundary value problems

Last time:

① The phenomenon of stiffness.

$$Tol_1 = Dh_1^{p+1}, \quad Tol_2 = Dh_2^{p+1} \longrightarrow \frac{h_1}{h_2} = \left(\frac{Tol_1}{Tol_2}\right)^{\frac{1}{p+1}}$$

focusing on linear systems as 1^{st} order approx to a general nonlinear system, using the toy equation

$$\dot{y} = \lambda y, \quad \lambda < 0$$

Euler stepping: $y_{n+1} = (1 + \lambda h) y_n$

$\longrightarrow$ stability $\longleftrightarrow$ $-1 < 1 + \lambda h < 1$

corrected $\boxed{-2 < \lambda h < 0}$

when $\lambda \gg 1$, h must satisfy a more restrictive constraint than ~~given~~ prescribed by tolerance

stability function $R(z)$

$$y_{n+1} = R(z)\, y_n, \quad z = \lambda h$$

$R(z)$ is a polynomial for multistep methods.

Dahlquist's theorem (1963) implied that explicit methods always has a stability function that is unbounded over $\lambda < 0$. — can never be $A(0)$-stable.

we should like for $\lambda < 0$ to be sufficient a ~~constrai~~ condition & the numerical method not to impose further constraints on h that competes with the one imposed by Tol.

this is $A(0)$-stability.

Consider Butcher tableaux :

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

backward / implicit Euler's method

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

trapezoidal rule

$$y_{n+1} = hf(x_{n+1}, y_{n+1}) + y_n$$

$$y_{n+1} = h \underline{\underline{\Lambda}}\, y_{n+1} + g(x_{n+1}) + y_n \qquad \underline{\underline{\Lambda}} \leq 0$$

$$\underbrace{(\underline{\underline{I}} - h\underline{\underline{\Lambda}})}_{A} y_{n+1} = g(x_{n+1}) + y_n$$

is always invertible.

② Numerical Differentiation

Finite elements methods.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x) - f(x-h)}{h} \qquad \text{when } f \in C^1$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

decline to take the limit:

$$D_h f(x) = \frac{f(x+h) - f(x)}{h} \qquad \text{(forward)}$$

$$\frac{f(x) - f(x-h)}{h} \qquad \text{(backwards)}$$

$$\delta_{2h} f(x) = \frac{f(x+h) - f(x-h)}{2h} \qquad \text{(central)}$$

or $$\delta_h f(x) = \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h}$$ on auxiliary grid of size $h/2$.

second derivatives:

$$D_h^2 f(x) = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}, \quad \delta_h^2 f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Thm Let $f \in C^4([x-h, x+h])$. Then,

$\exists\ \xi \in [x-h, x+h]$:

$$e_D^1(x;h) := f'(x) - D_h f(x) = -\frac{1}{2} f''(\xi) h$$

$\exists\ \eta \in [x-h, x+h]$:

$$e_\delta^1(x;h) := f'(x) - \delta_h f(x) = -\frac{1}{6} f^{(3)}(\eta) h^2$$

$\exists\ \xi \in [x-h, x+h]$:

corrected

$$e_\delta^2(x;h) := f''(x) - \delta_h^2 f(x) = -\frac{1}{12} f^{(4)}(\xi) h^2.$$

Pf. By Taylor's Expansion.

XXN. 1 Two Point Boundary Value Problems.

Consider the 2nd order linear ODE:

$$u'' + p(x) u' + q(x) u = r(x)$$

$$x \in [a,b], \quad u(a) = u_a, \quad u(b) = u_b$$

Apply FEM directly :

i) discretization of space :
$x_n = a + nh, \quad h = \frac{b-a}{N}, \quad n = 0, 1, \dots, N$

ii) At each grid pt. x_n, replace the derivatives :

$$r(x_n) = \frac{u(x_n+h) - 2u(x_n) + u(x_n-h)}{h^2} + p(x_n) \frac{u(x_n+h) - u(x_n-h)}{2h} + q(x_n)\, u(x_n) + O(h^2)$$

iii) Replacing $u(x_n)$ by U_n :

$$\begin{cases} h^2 r(x_n) = (U_{n+1} - 2U_n + U_{n-1}) \\ \quad + \frac{p(x_n)}{2} h\, (U_{n+1} - U_{n-1}) + q(x_n) U_n h^2 \end{cases}$$

iv) $\underline{U} = (U_0, \dots, U_N)'$

$$\underline{A}\,\underline{U} = \underline{b}$$

$\underline{b}$: depends on $r(x_n)$, h

$\underline{U}$: unknown to be solved.

$\underline{A}$: depends on $p(x_n)$, $q(x_n)$, h

$$\underline{A} = \begin{pmatrix} 1 & 0 & & & & \\ v_1 & d_1 & w_1 & & & \\ & v_2 & d_2 & w_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & v_{N-1} & d_{N-1} & w_{N-1} \\ & & & & 0 & 1 \end{pmatrix}$$

$$\underline{A} = \begin{pmatrix} 1 & 0 & & & & \\ v_1 & d_1 & w_1 & & & \\ & v_2 & d_2 & w_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & v_{N-1} & d_{N-1} & w_{N-1} \\ & & & & 0 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} v_i &= 1 - \tfrac{h}{2}\, p(x_i) \\ d_i &= -2 + h^2 q(x_i) \\ w_i &= 1 + \tfrac{h}{2}\, p(x_i) \end{aligned}\right\} \text{ for } i = 1, \dots, N-1$$

$$v_N = 0, \quad d_0 = d_N = 1, \quad w_0 = 0$$

$$(\underline{A})_{ij} = w_i\,\delta_{i\,j-1} + d_i\,\delta_{ij} + v_i\,\delta_{i\,j+1}$$

$j = 1, \dots, N-1$

Kronecker delta: $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$

$$\underline{b} = (u_a, h^2 r(x_1), \dots, h^2 r(x_{N-1}), u_b)^T$$

$$\underline{A}\,\underline{U} = \underline{b}$$

iv) invert $\underline{A}$ to solve for $\underline{U}$:

$$\underline{U} = \underline{A}^{-1} \underline{b}.$$

Example XXIV.1.

Consider $u'' + 2u' - 3u = 9x$ $\quad u(0) = u_0 = 1$

$u(1) = u_1 = e^{-3} + 2e - 5 \approx 0.48635$

$$u(x) = e^{-3x} + 2e^{x} - 3x - 2$$

$$h^2 9x_n = \frac{U_{n+1} - 2U_n + U_{n+1}}{\cancel{h^2}} + \cancel{2}h\frac{U_{n+1} - U_{n-1}}{\cancel{2h}} - 3U_n h^2$$

$U_0 = 1$, $U_N = 0.48635$ ← fixed.

Suppose now we specify the gradient at the boundary:

$$u'' + p(x)\,u' + q(x)\,u = r(x) \qquad x \in [a,b]$$

$$u'(a) = u'_a , \qquad u(b) = u_b$$

how shall we discretize the gradient BV?

suggestion 1: $u'_a = \dfrac{u(x_1) - u(x_0)}{h} + O(h)$

so $u'_a = \dfrac{U_1 - U_0}{h}$

suggestion 2: non-physical / false grid:

assume there is a pt. x_{-1} to the left of x_0,

i.e. $x_{-1} = a - h$.

assume an approx. value U_{-1} at x_{-1},

and write

$$u'_a = \frac{U_1 - U_{-1}}{2h}$$

so $U_{-1} = U_1 - 2h\,u'_a$

the false grid pt assumption changes our 0^{th} eq:

$$r(x_0) = \frac{U_1 - 2U_0 + U_{-1}}{h^2} + p(x_0)\frac{U_1 - U_{-1}}{2h} + q(x_0)\,U_0$$

inserting $U_{-1} = U_1 - 2h\,u_a'$, 0^{th} eq is:

$$r(x_0) = \frac{2U_1 - 2U_0 - 2h\,u_a'}{h^2} + p(x_0)\,u_a' + q(x_0)\,U_0$$

$$\underbrace{2}_{a_{01}}\,U_1 + \underbrace{(-2 + q(x_0)h^2)}_{a_{00}}\,U_0 = \underbrace{h^2 r(x_0) - p(x_0)\,u_a' h^2 + 2u_a' h}_{b_0}.$$

Example XXIV.2

same equation: $u'' + 2u' - 3u = 9x$

$$u'(0) = u_a' = -4$$
$$u(1) = u_b = 0.48635.$$

with $h = 0.25$

$-2 + q(x_0)h^2$

$$\underline{\underline{A}} = \begin{pmatrix} -2.1875 & 2 & & \\ 0.75 & -2.1875 & 1.25 & \\ & \ddots & \ddots & \ddots \\ & & 0 & 1 \end{pmatrix}$$