

Lecture XXIV

This time:

Last time:

- The phenomenon of stiffness.

Solving boundary value problems

$$\begin{aligned} \text{Tol}_1 &= Dh_1^{p+1} \\ \text{Tol}_2 &= Dh_2^{p+1} \end{aligned} \rightarrow \frac{h_1}{h_2} = \left(\frac{\text{Tol}_1}{\text{Tol}_2} \right)^{\frac{1}{p+1}}$$

focusing on linear systems as 1st order approx to a general nonlinear system, using the toy equation

$$\dot{y} = \lambda y, \quad \lambda < 0$$

$$\text{Euler stepping: } y_{n+1} = (1 + \lambda h) y_n$$

$$\rightarrow \text{stability} \Leftrightarrow 0 < 1 + \lambda h < 1$$

$$\Leftrightarrow -1 < \lambda h < 0$$

when $\lambda \gg 1$, h must satisfy a more restrictive constraint than ~~is~~ prescribed by tolerance.

Stability function $R(z)$

$$y_{n+1} = R(z) y_n, \quad z = \lambda h$$

$R(z)$ is a polynomial for multi-step methods.

Dahlquist's theorem (1963) implied that explicit methods always has a stability function that is unbounded over $\lambda < 0$. — can never be $A(\infty)$ -stable.

We should like for $\lambda < 0$ to be sufficient a ~~constraint~~ condition for the numerical method not to impose further constraints on h that competes with the one imposed by Tol. This is $A(\infty)$ -stability.

Considérer Butcher Tableaux :

1	1
1	1

0	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$
1	$\frac{1}{2}$	$\frac{1}{2}$

backward / implicit Euler's method

trapezoidal rule

$$y_{n+1} = h f(x_{n+1}, y_{n+1}) + y_n$$

$$y_{n+1} = h \Delta y_{n+1} + g(x_{n+1}) + y_n \Delta \leq 0$$

$$(1 - h \Delta) y_{n+1} = g(x_{n+1}) + y_n$$

Δ is always invertible.

② Numerical Differentiation

finite elements methods.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x) - f(x-h)}{h} \quad \text{when } f \in C^1 \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h} \end{aligned}$$

decide to take the limit:

$$D_h f(x) = \frac{f(x+h) - f(x)}{h} \quad (\text{forward})$$

$$\delta_h f(x) = \frac{f(x) - f(x-h)}{h} \quad (\text{backwards})$$

$$\delta_{2h} f(x) = \frac{f(x+h) - f(x-h)}{2h} \quad (\text{central})$$

$$\text{or } \delta_h f(x) = \frac{f(x + \frac{h}{2}) - f(x - \frac{h}{2})}{h} \quad \text{on auxiliary grid of size } \frac{h}{2}.$$

second derivatives:

$$D_h^2 f(x) = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}, \quad \delta_h^2 f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Thm. Let $f \in C^4([x-h, x+h])$, Then,

$\exists \xi \in [x-h, x+h]$:

$$e_D^1(x; h) := f'(x) - D_h f(x) = -\frac{1}{2} f''(\xi) h$$

$\exists \eta \in [x-h, x+h]$:

$$e_S^1(x; h) := f(x) - S_h f(x) = -\frac{1}{6} f^{(3)}(\eta) h^2$$

$\exists \zeta \in [x-h, x+h]$:

$$e_S^2(x; h) := f''(x) - S_h^2 f(x) = -\frac{1}{12} f^{(4)}(\zeta) h^4$$

Pf. By Taylor's Expansion.

XXN. 1 Two Point Boundary Value Problems.

Consider the 2nd order linear ODE:

$$u'' + p(x) u' + q(x) u = r(x)$$

$$x \in [a, b], \quad u(a) = u_a, \quad u(b) = u_b$$

Apply FEM directly:

discretization of space:

$$i) \quad x_n = a + nh, \quad h = \frac{b-a}{N}, \quad n = 0, 1, \dots, N$$

ii) At each grid pt. x_n , replace the derivatives:

$$r(x_n) = \frac{u(x_n+h) - 2u(x_n) + u(x_n-h)}{h^2}$$

$$+ p(x_n) \frac{u(x_n+h) - u(x_n-h)}{2h}$$

$$+ q(x_n) u(x_n) + O(h^2)$$

iii) Replacing $u(x_n)$ by U_n :

$$\boxed{h^2 r(x_n) = (U_{n+1} - 2U_n + U_{n-1})}$$

$$+ \frac{p(x_n)}{2} h (U_{n+1} - U_{n-1}) + q(x_n) U_n h^2$$

$$\text{iv) } \underline{U} = (U_0, \dots, U_N)^T$$

$$\underline{A} \underline{U} = \underline{b} \quad \begin{array}{l} \text{depends on } r(x_n), h \\ \text{q: unknown to be solved.} \\ \text{depends on } p(x_n), q(x_n), h \end{array}$$

$$\underline{A} = \left(\begin{array}{ccc|c} 1 & 0 & & \\ v_1 & d_1 & w_1 & \\ & v_2 & d_2 & w_2 \\ & & \ddots & \vdots \\ & & v_{N-1} & d_{N-1} & w_{N-1} \\ & & & 0 & \end{array} \right)$$

$$\underline{A} = \left(\begin{array}{ccc|c} 1 & 0 & & \\ \cancel{v_1} & \cancel{d_1} & \cancel{w_1} & \\ \cancel{v_2} & \cancel{d_2} & \cancel{w_2} & \\ \vdots & \vdots & \vdots & \vdots \\ \cancel{v_{N-1}} & \cancel{d_{N-1}} & \cancel{w_{N-1}} & 0 \end{array} \right)$$

$$\left. \begin{array}{l} v_i = 1 - \frac{h}{2} p(x_i) \\ d_i = -2 + h^2 q(x_i) \\ w_i = 1 + \frac{h}{2} p(x_i) \end{array} \right\} \text{for } i = 1, \dots, N-1$$

$$v_N = 0, \quad d_0 = d_N = 1, \quad w_0 = 0$$

$$(\underline{A})_{ij} = w_i \delta_{i,j+1} + d_i \delta_{ij} + v_i \delta_{i,j+1}$$

Kronecker delta : $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases} \quad j = 1, \dots, N-1$

$$\underline{b} = (u_a, h^2 r(x_1), \dots, h^2 r(x_{N-1}), u_b)^T$$

$$\underline{A} \underline{U} = \underline{b}$$

iv) invert \underline{A} to solve for \underline{U} :

$$\underline{U} = \underline{A}^{-1} \underline{b}.$$

Example XIV.1

Consider

$$u'' + 2u' - 3u = 9x$$

$$u(0) = u_a = 1$$

$$u(1) = u_b = e^{\frac{3}{2}} + 2e - 5$$

$$u(x) = e^{-3x} + 2e^x - 3x - 2$$

$$0.48635$$

$$h^2 g_{x_n} = \frac{U_{n+1} - 2U_n + U_{n-1}}{h^2} + 2h \frac{U_{n+1} - U_{n-1}}{2h} - 3U_n h^2$$

$$U_0 = 1, \quad U_N = 0.48635 \quad \leftarrow \text{fixed.}$$

Suppose now we specify the gradient at the boundary:

$$u'' + p(x)u' + q(x)u = r(x) \quad x \in [a, b]$$

$$u'(a) = u'_a, \quad u(b) = u_b$$

how shall we discretize the gradient BV?

$$\text{suggestion 1: } u'_a = \frac{u(x_1) - u(x_0)}{h} + O(h)$$

$$\text{so } u'_a = \frac{U_1 - U_0}{h}$$

suggestion 2: non-physical / false grid:

assume there is a pt. x_{-1} to the left of

$$\text{i.e. } x_{-1} = a - h. \quad x_0,$$

assume an approx. value U_{-1} at x_{-1} ,
and write

$$u'_a = \frac{U_1 - U_{-1}}{2h}$$

$$\text{so } U_{-1} = U_1 - 2h u'_a$$

The false grid pt assumption changes our eq:

$$r(x_0) = \frac{U_1 - 2U_0 + U_{-1}}{h^2} + p(x_0) \frac{U_1 - U_{-1}}{2h} + q(x_0) U_0$$

Inserting $U_{-1} = U_1 - 2hu_a'$, 0th eq is:

$$r(x_0) = \frac{2U_1 - 2U_0 - 2hu_a'}{h^2} + p(x_0) u_a' + q(x_0) U_0$$

a_{01} q_{00} $+ 2u_a' h.$

b_0

Example XXIV. 2

same equation: $u'' + 2u' - 3u = g(x)$

$$u'(0) = u_a' = -4$$

$$u(1) = u_b = 0.48635$$

with $h = 0.25$

$$A = \begin{pmatrix} -2.1875 & 2 & & \\ 0.75 & -2.1875 & 1.25 & \\ & \ddots & \ddots & \ddots \\ & & & 0 & 1 \end{pmatrix}$$

, $2 + q(x_0)h^2$