

Lecture XXXII

This time:

- ① Step size control
- ② Runge-Kutta methods
- ③ Stiff equations (begin)

Last time:

- ① we introduced Heun's method for

$$y'(x) = f(x, y(x)) .$$

$$u_{n+1} = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2} (f(x_n, y_n) + f(x_n, u_{n+1}))$$

~~Error analysis~~

(A)

$$\Phi(x, y; h) = \frac{1}{2} [f(x, y) + f(x+h, y+h)] f(x, y)$$

$$y_{n+1} = y_n + h \Phi(x_n, y_n; h) \quad \text{B Lipschitz in } y$$

(B)

$$d_{n+1} = y(x_{n+1}) - (y(x_n) + h \Phi(x_n, y(x_n); h))$$

$$\frac{dy}{dx} \quad \text{local truncation error} \quad |d_{n+1}| \leq h^3$$

(A), (B) → together imply that $|e_n| \sim h^2$

$$(h \mapsto h/2) \rightarrow (e_n \mapsto e_n/4)$$

- ② Stepsize control, had following algorithm:

For some chosen step size h_n , and some known values

(x_n, y_n) , let an approximate error $\|e_{n+1}(h_n)\|$ be defined

If $\|e_{n+1}\| \leq \text{Tol}$ → then accept h_n ,
set $x_{n+1} = x_n + h_n$

else if $\|e_{n+1}\| > \text{Tol}$ → choose a revised, smaller \hat{h}_n
to replace h_n .

compute $\|e_{n+1}(\hat{h}_n)\|$
repeat from beginning.

↑

XXXII.1 Stepsize Control II

(i) What is e_{n+1} ?

(ii) how do we choose the revised \hat{h}_n ~~as~~ efficiently?

i) What is le_{n+1} ?

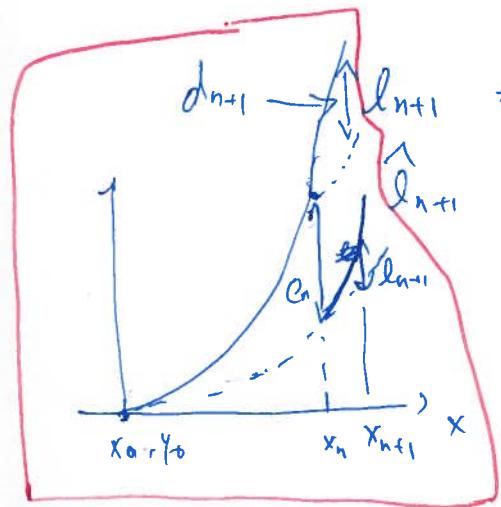
Let $\underline{\Phi}, \hat{\Phi}$ be two increment functions for two 1-step methods of orders $p, p+1$, resp.

Starting from a common (x_n, y_n) ,

$$y_{n+1} = y_n + h \underline{\Phi}(x_n, y_n; h)$$

$$\hat{y}_{n+1} = y_n + h \hat{\Phi}(x_n, y_n; h)$$

Let $y(x_{n+1}; x_n, y_n)$ be the exact solution over $[x_n, x_{n+1}]$ starting at (x_n, y_n)



$$\begin{aligned} d_{n+1} - l_{n+1} &= y(x_{n+1}; x_n, y_n) - y_{n+1} \\ &= y(x_{n+1}; x_n, y_n) - \hat{y}_{n+1} \end{aligned}$$

If it is sufficient to assume that

$$|l_{n+1}| \lesssim h^{p+1}$$

$$|\hat{l}_{n+1}| \lesssim h^{p+2}$$

$$y(x_{n+1}; x_n, y_n) - y_{n+1} \approx l_{n+1} = \underline{\Phi}(x_n, y_n) h^{p+1} + O(h^p)$$

$$y(x_{n+1}; x_n, y_n) - \hat{y}_{n+1} \approx \hat{l}_{n+1} = O(h^{p+2})$$

so as $h \rightarrow 0$,

$$\hat{y}_{n+1} - y_n = \underline{\Phi}(x_n, y_n) h^{p+1} + O(h^{p+2})$$

approximate local error $\underline{(le)}_{n+1} := \underline{\Phi}(x_n, y_n) h^{p+1} + O(h^{p+2})$

(ii) How can we choose \hat{h}_n efficiently?

If $|l_{e_{n+1}}(h_n)| > Tol$.

then we have found revised \hat{h}_n .

$$|l_{e_{n+1}}| \approx D_n h_n^{p+1}$$

$$Tol \approx D_n \hat{h}_n^{p+1} \quad \text{take a ratio!}$$

if h_n isn't good enough,

$$\left| \frac{Tol}{l_{e_{n+1}}(h_n)} \right| \approx \left(\frac{\hat{h}_n}{h_n} \right)^{p+1}$$

$$\text{Rearrange : } \hat{h}_n \leq h_n \cdot \left(\frac{Tol}{l_{e_{n+1}}(h_n)} \right)^{\frac{1}{p+1}}.$$

$$\rightarrow |l_{e_{n+1}}(\hat{h}_n)| \leq Tol.$$

XXII.2 Runge-Kutta Methods

In general a Runge Kutta method can be written as

$$\begin{aligned}
 & \left. \begin{array}{l} k_1 = f(x_n, y_n) \\ k_2 = f(x_n + c_2 h, y_n + h a_{21} k_1) \\ k_3 = f(x_n + c_3 h, y_n + h(a_{31} k_1 + a_{32} k_2)) \\ \vdots \\ k_s = f(x_n + c_sh, y_n + h \sum_{j=1}^s a_{sj} k_j) \\ y_{n+1} = y_n + h \sum_{j=1}^s b_j k_j \end{array} \right\} \\
 & \text{Euler's method} \quad s=2, c_2=1, a_{21}=1, b_1=\frac{1}{2}, b_2=\frac{1}{2} \\
 & \text{Heun's method} \quad \text{for trapezoid}
 \end{aligned}$$

$$\text{trapezoidal rule : } \underline{y_{n+1}} = y_n + \frac{h}{2} (f(x_n, y_n) + f(x_{n+1}, \underline{y_{n+1}}))$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + h, y_n + \frac{h}{2} (k_1 + \underline{k_2}))$$

$$y_{n+1} = y_n + \frac{h}{2} (k_1 + \underline{k_2})$$

If $a_{kj} \neq 0$ for ~~$k \leq j$~~ then ~~the RK~~ is implicit

~~Explained~~ RK

$$\begin{cases} k_i = f(x_n + c_i h, y_n + h a_{ij} k_j) \\ y_{n+1} = y_n + h b_j k_j \end{cases} \rightarrow k_i = f(x_n, y_n)$$

Convex combination

Def (Consistency)

A numerical method is consistent with the differential equation

$$y'(x) = f(x, y(x))$$

If the local truncation error is such that $\forall \varepsilon > 0, \exists h(\varepsilon) > 0$:

$$0 < h < h(\varepsilon) \rightarrow \frac{|d_n|}{h} < \varepsilon \quad \frac{d_n}{h} \leq M \mathcal{O}(1)$$

at any pair of points $(x_n, y(x_n)), (x_{n+1}, y(x_{n+1}))$.

If local trunc. error $\rightarrow 0$ as $h \rightarrow 0$

then global error $\rightarrow 0$

Assuming f is continuous in its arg., as $h \rightarrow 0$

In order to remain at ~~$x_n = X$~~ $x_n \in [x_0, x_n]$

we have to take $n \rightarrow \infty$

$$d_{n+1} := y(x_{n+1}) - (y(x_n) + h \Phi(x_n, y(x_n); h))$$

$$\frac{d_{n+1}}{h} = \underbrace{\frac{y(x_{n+1}) - y(x_n)}{h}}_{\Phi(x_n, y(x_n); 0)} \neq \Phi(x_n, y(x_n); h)$$

$$0 = \lim_{n \rightarrow \infty} \frac{d_{n+1}}{h} = y'(x) \neq \Phi(x, y(x); 0)$$

$$\boxed{f(x, y(x)) = \Phi(x, y(x); 0)}$$

for RK :- $f(x, y(x)) = \sum b_i f(x, y(x))$

$$\sum b_i = 1$$

$$\frac{1}{c_i} \sum_j a_{i,j} = 1$$

c_1	a_{11}	a_{12}	\dots	a_{1s}
c_2	a_{21}	a_{22}	\dots	a_{2s}
\vdots	\vdots	\vdots	\ddots	\vdots
c_s	a_{s1}	a_{s2}	\dots	a_{ss}
b_1	b_2	\dots	b_s	

pth order accurate RK methods

P	
1	$\sum b_i = 1$
2	$\sum b_i c_i = \frac{1}{2}$ and above
3	$\begin{cases} \sum b_i c_i^2 = \frac{1}{3} \\ \sum b_i a_{ij} c_j = \frac{1}{6} \text{ and above} \end{cases}$
4	$\begin{cases} \sum b_i c_i^3 = \frac{1}{4} \\ \sum b_i a_{ij} c_j^2 = \frac{1}{8} \\ \sum b_i c_i a_{ij} c_j = \frac{1}{8} \text{ and above} \\ \sum b_i a_{ij} a_{jk} c_k = \frac{1}{24} \end{cases}$

- Classical 4th order RK method (RK4)

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h k_1)$$

$$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h k_2)$$

$$k_4 = f(x_n + h, y_n + h k_3)$$

$$y_{n+1} = y_n + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$\begin{array}{c|cc} 0 & 0 \\ \hline 1 & \end{array}$$

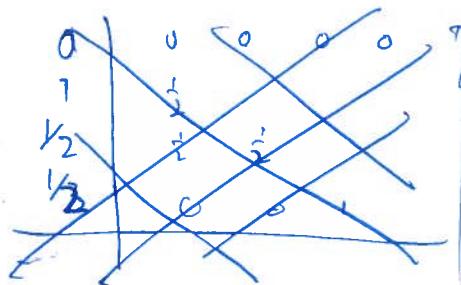
Euler

$$\begin{array}{c|ccc} 0 & 0 & 0 \\ \hline 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Heun

$$\begin{array}{c|ccc} 0 & 0 & 0 \\ \hline 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

trapezoidal Rule.



$$\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 \\ \hline \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline 1 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

RK4

XXII , 3 Stiff ODEs .

$$\| (\text{le})_{n+1}(h_n) \| \leq T_0 \approx D h_n^{p+1}$$

$$T_0 l_1, T_0 l_2$$

$$T_0 l_1 = D h_1^{p+1} \quad T_0 l_2 = D h_2^{p+2}$$

\uparrow
not $\frac{n+1}{n+2}$

$$\frac{N_2}{N_1} = \frac{\frac{h-a}{N_1}}{\frac{b-a}{N_2}} = \frac{h_1}{h_2} \approx \left(\frac{T_0 l_1}{T_0 l_2} \right)^{\frac{1}{p+1}} \approx \frac{N_2}{N_1}$$