

Lecture XXI

Last time

- ① Finished up Newton-Raphson method for systems of equations:

By Taylor's theorem, at a root $\underline{x} \in \mathbb{R}^d$

$$0 = \underline{f}(\underline{x}) = \underline{f}(\underline{x}) + \nabla \underline{f}(\underline{x}) \cdot (\underline{x} - \underline{x}) + \mathcal{O}(|\underline{x} - \underline{x}|^2)$$

where $(\nabla \underline{f})_i^j = \partial_j f^i$ j is upper index, not a power

is the Jacobian matrix

dropping higher order terms; we expect

$$0 = \underline{f}(\underline{x}_k) + \nabla \underline{f}(\underline{x}_k) \cdot (\underline{x}_k - \underline{x}_{k+1}),$$

rearranging,

$$\underline{x}_{k+1} = - \left(\nabla \underline{f} \Big|_{\underline{x}_k} \right)^{-1} \left(\underline{f}(\underline{x}_k) + \underbrace{\nabla \underline{f}(\underline{x}_k) \underline{x}_k}_{\text{inverse matrix}} \right)$$

- ② Euler's method on a system of first order ODEs.

$$y'(x) = \underline{f}(x, y(x)).$$

Again by Taylor's theorem,

$$\begin{aligned} y(x_0 + h) &= y(x_0) + h y'(x_0) + \mathcal{O}(h^2) \\ &= y(x_0) + h \underline{f}(x_0, y(x_0)) + \underline{\underline{\mathcal{O}(h^2)}} \end{aligned}$$

discretizing space into $x_n = nh$,

(Forward Euler's method)

$$y_{n+1} = y_n + h f(x_n, y_n),$$

- ③ Error analysis for Euler's method:

we looked at Global Error: $e_n = y(x_n) - y_n$

local error: $d_{n+1} = y(x_{n+1})$

$$- \left(y(x_n) + h f(x_n, y(x_n)) \right)$$

1 step of Euler's method from exactly known $y(x_n)$.

Euler's method is 1st order

if $(h \rightarrow h/2)$, then $(e_n \rightarrow e_n/2)$

$$|d_{n+1}| = \mathcal{O}(h^2), \quad |e_{n+1}| \leq |e_n| (1+L) + |d_{n+1}| \rightarrow |e_n| \lesssim h(b-a) \leq e h(b-a).$$

④ General 1-step methods :

$$y_{n+1} = y_n + \Phi(x_n, y_n; h)$$

↑ INCREMENT FUNCTION
only ~~works~~ at last step.

Thm.

if (i) Φ is Lipschitz in its 2nd argument

$\exists L < \infty$:

$$|\Phi(x, y; h) - \Phi(x, z; h)| \leq L|y - z|$$

different!

(ii) the local error is controlled thus :

$$|d_{n+1}| \lesssim h^{p+1}$$

then, the one step method (Φ) is order p ,

i.e. $|e_n| = |y(x_n) - y_n| \lesssim h^p$.

(Proved this for Euler's method with $p=1$).

⑤ ~~XX~~ .1 Heun's Method. for $y' = f(x, y)$.

$$(h \mapsto h/2) \rightarrow (e_n \mapsto e_n/2^p)$$

twice as many iterations/computations

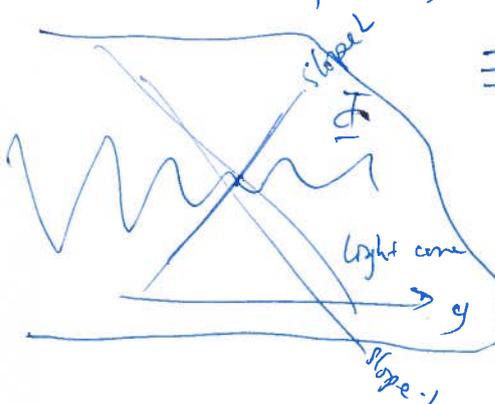
By the fund. thm. of calculus

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + \int_{x_n}^{x_{n+1}} y'(r) dr \\ &= y(x_n) + \int_{x_n}^{x_{n+1}} \underbrace{f(r, y(r))}_{f(x, y)} dr \end{aligned}$$

Use now the trapezoidal rule to approximate ~~this~~ integral:

$$(x_{n+1} - x_n = h) \quad y(x_{n+1}) \approx y(x_n) + \frac{h}{2} (f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1})))$$

$$y_{n+1} = y_n + \frac{h}{2} (f(x_n, y_n) + f(x_{n+1}, \underline{y_{n+1}}))$$



introduce an auxiliary step:

$$\begin{cases} u_{n+1} = y_n + h f(x_n, y_n) & \text{(Euler stepping)} \\ y_{n+1} = y_n + \frac{h}{2} (f(x_n, y_n) + f(x_{n+1}, u_{n+1})) \end{cases}$$

HEUN'S METHOD

also can be written as

$$\begin{cases} k_1 = f(x_n, y_n) \\ k_2 = f(x_{n+1}, y_n + h k_1) \\ y_{n+1} = y_n + \frac{h}{2} (k_1 + k_2) \end{cases}$$

Using the notation of (4)

$$y_{n+1} = y_n + \Phi(x_n, y_n; h)$$

Heun's method: $\Phi(x, y, h) = \frac{1}{2} [f(x, y) + f(x+h, y+h f(x, y))]$

Heun's method is 2nd order

$$(h \mapsto h/2) \longrightarrow (e_n \mapsto e_n/4)$$

Error analysis for Heun's method

in order to get $|R_n| \approx h^p$

it is sufficient to show/have

- (A) $\Phi(x, y, h)$ Lipschitz in the 2nd argument
- (B) $|d_{n+1}| \approx h^{p+1}$

(A) further assume that f is Lipschitz in its second argument

$$\exists L < \infty : |f(x, y) - f(x, z)| \leq L |y - z|$$

Then $| \Phi(x, y) - \Phi(x, z) |$

$$= \frac{1}{2} | f(x, y) + f(x+h, y+h f(x, y)) - f(x, z) - f(x+h, z+h f(x, z)) |$$

Δ -ineq.:

$$\leq \frac{1}{2} | f(x, y) - f(x, z) | + \frac{1}{2} | f(x+h, y+h f(x, y)) - f(x+h, z+h f(x, z)) |$$

Lipschitz cond.

$$\leq \frac{1}{2} L |y - z| + \frac{1}{2} L | (y + h f(x, y)) - (z + h f(x, z)) |$$

$$\Delta\text{-ineq} : \leq \frac{1}{2} L |y-z| + \frac{1}{2} L |y-z| + \frac{1}{2} Lh |f(x,y) - f(x,z)|$$

$$\text{Lipschitz cond.} \leq \frac{1}{2} L |y-z| + \frac{1}{2} L |y-z| + \frac{1}{2} L^2 h |y-z|$$

$$= \underline{\left(L + \frac{1}{2} L^2 h \right) |y-z|}$$

↑ Lipschitz constant for Φ (in 2nd arg.)

(B)

$$d_{n+1} = y(x_{n+1}) - \left(y(x_n) + \Phi(x_n, y(x_n); h) \right)$$

$$= y(x_{n+1}) - \left[y(x_n) + \frac{h}{2} \left[f(x_n, y(x_n)) + f(x_{n+1}, y(x_n) + h f(x_n, y(x_n))) \right] \right]$$

$$y(x_{n+1}) = y(x+h) = y(x) + h y'(x) + \frac{h^2}{2} y''(x) + \frac{h^3}{6} y^{(3)}(x) + \mathcal{O}(h^4)$$

" $f(x, y(x))$

$$f_x, f_y, f_{xy}, f_{xx}, f_{yy}$$

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \dots \text{ et cetera.}$$

assume that these are evaluated at $(x_n, y(x_n))$

$$y'(x_n) = f$$

$$y''(x_n) = f_x + f_y y'(x_n) = f_x + f_y f$$

$$y^{(3)}(x_n) = f_{xx} + 2f_{xy} y'(x_n) + f_{yy} (y'(x_n))^2 + f_y f_x + f_y f_y y'(x_n)$$

Putting all these derivatives into the expression for d_{n+1}

$$d_{n+1} = \frac{h^3}{12} \left(-f_{xx} - 2f_{xy} f - f_{yy} f^2 + 2f_y f_x + 2(f_y)^2 f \right) + \mathcal{O}(h^4)$$

$$|d_{n+1}| \lesssim h^3$$

By previous thm. Lipschitz Φ & $|d_{n+1}| \lesssim h^3$

$$\rightarrow |e_n| \lesssim h^2$$

XXI, 2 Step size control

$(le)_{n+1}$ $\xleftarrow{\text{stopped } h}$ from n^{th} iterate.

Using some $\|\cdot\|$,

if $\|(le)_{n+1}\| < Tol \rightarrow$ accept h_n

$$x_{n+1} = x_n + h_n$$

$$y_{n+1} = y_n + \Phi(x_n, y_n; h_n).$$

else replace h_n by a smaller, \hat{h}_n ,
revised

repeat calculation of $(n+1)$ st step

& $(n+1)^{\text{st}}$ $\|(le)_{n+1}\|$.

Let $\bar{\Phi}, \hat{\Phi}$ be two increment functions of two different step methods of orders p and $p+1$, resp.

$$y_{n+1} = y_n + \bar{\Phi}(x_n, y_n; h)$$

$$\hat{y}_{n+1} = y_n + \hat{\Phi}(x_n, y_n; h)$$