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TMA4130
Matematikk 4N
Spring 2018

Exercise set 9

The theory and the codes are taken from the Jupyter notebook on *Polynomial Interpolation* and *Numerical Quadrature*.

Exercises supposed to be done by hand are marked with an (H).

Exercises in which you are supposed to use/modify code in the Jupyter notebook is marked with a (J).

For Jupyter-exercises, hand in a screen-dump of the relevant cell with output.

- 1 a) (H) Given the points

$$\begin{array}{c|ccc} x_i & -2 & 1 & 6 \\ \hline y_i & 1 & 2 & 3 \end{array}$$

Set up the table of divided differences, and write down the second order interpolation polynomial in the Newton form.

- b) (H) Find the interpolation polynomial interpolating the points from a) and one extra point, $x_3 = -3/4$ and $y_3 = 3/2$.

Inverse interpolation This is a strategy that can be used to find approximations to solutions of an equation $f(x) = 0$.

Given a sufficiently continuous function $f(x)$, with a root r in some interval $[a, b]$. Choose this interval sufficiently small to ensure f to be strict monotonically increasing or decreasing at $[a, b]$. Under these conditions f is invertible on the interval, in the sense that

$$y = f(x) \quad \Rightarrow \quad x = f^{-1}(y).$$

In particular:

$$f(r) = 0 \quad \Rightarrow \quad r = f^{-1}(0).$$

The strategy is then: Choose $n + 1$ distinct data points (x_i, y_i) in the interval $[a, b]$ and find the interpolation polynomial $p_n(y)$ satisfying

$$p_n(y_i) = x_i.$$

In this case, $p_n(y) \approx f^{-1}(y)$, and $r \approx p_n(0)$.

- c) (H) Let $f(x) = x^2 - 3$, and $[a, b] = [1, 3]$. As nodes, choose $x_0 = 1, x_1 = 2, x_2 = 3$ and use the idea outlined above to find an approximation to the solution of $f(x) = 0$. How close to the exact solution is the approximation?

To get a better approximation, add the node $x_3 = 3/2$. Will this provide a better result?

Hint: Use the results from **a**).

2 Consider the integral

$$\int_1^3 e^{-x} dx.$$

- a) (H) Find numerical approximations to the integral using Simpson's method over 1 and 2 intervals, that is $S_1(1, 3)$ and $S_2(1, 3)$. Find an error estimate for $S_2(1, 3)$, and compare with the real error.
- b) (H) Find the number of intervals m that guarantees that the approximation of the integral by the composite (sammensatt) Simpson's method is less than 10^{-8} .
NB! $h = (b - a)/(2m)$.
- c) (J) Find the numerical approximation of the integral by using the function `simpson`, and m from point **b**). Verify that the error is less than 10^{-8} .

3 The aim of this exercise is to repeat the exercise done on Simpson's method in the note, but now with a Gauss–Legendre quadrature. Which is described at the end of the note.

- a) (H) Find the Gauss–Legendre quadrature over the interval $[-1, 1]$ with $m = 3$.
- b) (H) Confirm that the quadrature has degree of precision 5.
- c) (H) Transfer the quadrature over to some arbitrary interval $[a, b]$.
Try to express the quadrature more elegantly, e.g., by introducing $c = (b + a)/2$ and $h = (b - a)/2$, and the nodes in terms of c and h .
Use it to find an approximation to $\int_1^3 e^{-x} dx$.
What is the error?

The error of this Gauss–Legendre quadrature over one interval is given by

$$E(a, b) = \int_a^b f(x) dx - Q(a, b) = \frac{(b - a)^7}{2016000} f^{(6)}(\eta), \quad \eta \in (a, b).$$

- d) (H) Use this to find an error expression for the composite Gauss–Legendre quadrature

$$Q_m(a, b) = \sum_{k=0}^{m-1} Q(X_k, X_{k+1}),$$

where $Q(X_k, X_{k+1})$ is the basic quadrature over the interval $[X_k, X_{k+1}]$, $X_k = a + kH$, $k = 0, 1, \dots, m$ and $H = (b - a)/m$.

Based on this expression, find the number of intervals m that guarantees that the error in the composed method is less than 10^{-8} when applied to the problem in point **b**). Compare with the similar result for Simpson's method.

- e) (H) Based on $Q_1(a, b)$ and $Q_2(a, b)$, find an error estimate for $Q_2(a, b)$ (you may assume that $f^{(6)}(x)$ is almost constant over the interval).

- f) (J – Optional) Write a function `gauss_basis` (similar to `simpson_basis`) implementing the Gauss–Legendre quadrature with error estimate.

Check your function by ensuring that all polynomials of degree 5 or less are exactly integrated.

Use the function to confirm your result from point c). Compare the exact error and the error estimate.

4 Chebyshev Points (H, optional).

We shall derive the Chebyshev polynomials and their zeros

- a) Show that $T_n(x) = \cos(n \arccos(x))$ for $x \in [-1, 1]$ satisfies the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

That is, the Chebyshev polynomials $T_0(x)$ and $T_1(x)$ imply all other polynomials by means of this recurrence formula. (Hint: Find $T_0(x)$ and $T_1(x)$ by inspection and use

$$\cos((n+1)\vartheta) + \cos((n-1)\vartheta) = 2\cos(\vartheta)\cos(n\vartheta)$$

with $\vartheta = \arccos(x)$.)

- b) Derive the formula for the roots of $T_{n+1}(x)$. (Hint: These are the Chebyshev points.)
- c) Show that for Chebyshev polynomials written in the form

$$T_{n+1}(x) = c_{n+1}x^{n+1} + c_nx^n + \cdots + c_1x + c_0$$

the leading coefficient satisfies $c_{n+1} = 2^n$. Use this to prove that the polynomial

$$\omega_{\text{cheb}}(x) = \prod_{i=0}^n (x - x_i^{\text{cheb}})$$

satisfies

$$|\omega_{\text{cheb}}(x)| \leq \frac{1}{2^n} \text{ for all } x \in [-1, 1].$$

- d) Compute the integral

$$\int_{-1}^1 \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx.$$

One says that the Chebyshev polynomials form an orthogonal set on the interval $[-1, 1]$ with a so-called weight function $1/\sqrt{1-x^2}$ (Hint: recall the computation of the integral $\int_0^\pi \cos(n\lambda)\cos(m\lambda)d\lambda$ for integers m, n .)