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Exercise set 12

1 **a)** (H) Find an approximation to $f'(x)$ based on the values of f in x, $x-h$ and $x-2h$. The approximation should be of as high order as possible. Find an expression for the error (a Taylor-expansion of the error with the first non-zero term written out is sufficient).

b) Use this to construct a time-stepping method for solving the ordinary differential equation

$$y' = f(x, y)$$

assuming that $y(0)$ and $y(h)$ are known. The method should be of the form

$$y_{n+1} = \alpha y_n + \beta y_{n-1} + h\gamma f(x_{n+1}, y_{n+1}).$$

where $x_n = nh$ and $y_n \approx y(x_n)$.

c) Use the method above to find an approxion to the solution $y(0.4)$ of the ODE

$$y' = xy,$$

when $y(0) = 0.5$ and $y(0.2) = .51010067$ are given.

2 (H) Find the interpolation polynomial $p_2(x)$ of the function f in some given points x_0, $x_0 + \theta h$ and $x_0 - h$. Use this to find approximations to $f'(x_0)$ and $f''(x_0)$.

What are the orders of the two approximations?

3 In this exercise, you are strongly recommended to work out the details of the difference schemes from the difference formulas for the derivatives, and nothing else.

Given the two point boundary value problem:

$$u_{xx} - 2u_x + u = 0, \quad 0 \le x \le 1, \qquad u(0) = 2, \qquad u_x(1) = 0$$

The exact solution is

$$u(x) = (2 - x)e^x.$$

a) Set up a finite difference scheme for this problem, using central differences. For the right hand boundary, use the idea of a false boundary and central differences. Use $\Delta x = 1/N$ as the grid size, and let $x_i = i\Delta x$, $i = 0, 1, \ldots, N$.

b) (H) Let $N = 2$ and use the above formula to find approximations $U_i \approx u(x_i)$, $i = 1, 2$. (That is: Set up the system of equations, and solve it). Compare with the exact solution.

c) (J) Modify the code `Example 1, BVP` in the note, and solve the problem numerically. Use $N = 10, 20, 40$ in your simulation. For each N, write down the error

$$e(h) = \max_{i=0,\dots,N} |u(x_i) - U_i|.$$

What can you deduce about the order of the scheme from this experiment?

d) (J) Repeat point **a)** and **c)**, but this time by using a backward difference (and no false boundary) to approximate the boundary condition $u_x(1) = 0$. Compare the error and the order of the scheme with your previous results.

e) (H) Assume that the right boundary condition is changed to

$$u_x + u = 0 \qquad \text{at } x = 1.$$

What will the difference equation in the boundary point $x = 1$ be in this case?

4 Given the time-dependent PDE:

$$u_t = u_{xx} - 2u_x + u, \qquad u(0,t) = 2, \quad u_x(1,t) = 0.$$

with initial values

$$u(x,0) = 2(1-x)^2.$$

a) (H) Do a semi-discretization of the PDE by using central differences for the approximations in the x-directions, with gridsize $\Delta x = 1/M$ for some M.
Choose $M = 2$ and set up the ODE system in this case.

b) (H) You now want to solve the system of ODEs from point **a)** (with arbitrary M) by the trapezoidal rule. Set up the linear system of equations that has to be solved for each time step, using $\Delta t = 0.5$
Again, set up the system for $M = 2$ and perform the first step.

c) (J) Modify the code for the heat equation in the PDE note, in order to solve this equation. Test your code with $M = 2$ and one step with $\Delta x = 0.5$ to verify the solution in point **b)**.
Solve the problem to $t_{end} = 1$ using $\Delta x = \Delta t = 0.05$ (but you may try other values as well).