Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions to exercise set
Final exam

1 **a)** Compute Laplace transform of

$$f(t) = te^t.$$

- By definition, we have

$$F(s) = \int_0^\infty te^t e^{-st}\, dt = \int_0^\infty te^{-(s-1)t}\, dt = \mathcal{L}(t)(s-1) = \frac{1}{(s-1)^2}.$$

b) Compute the inverse Laplace transform $\mathcal{L}^{-1}(F)(t)$ of the following function

$$F(s) := \frac{s+3}{s(s-1)(s+2)}.$$

(Hint: you can use partial fraction decomposition).

- Notice that

$$\frac{A}{2s} + \frac{B}{3(s-1)} + \frac{C}{6(s+2)} = \frac{3A(s-1)(s+2) + 2Bs(s+2) + Cs(s-1)}{6s(s-1)(s+2)},$$

comparing the coefficients gives

$$3A + 2B + C = 0,\ 3A + 4B - C = 6,\ -6A = 18.$$

Thus

$$A = -3,\ B = 4,\ C = 1,$$

and therefore

$$\mathcal{L}^{-1}(F)(t) = -\mathcal{L}^{-1}(\frac{3}{2s})(t) + \mathcal{L}^{-1}(\frac{4}{3(s-1)})(t) + \mathcal{L}^{-1}(\frac{1}{6(s+2)})(t).$$

This then yields

$$f(t) = -\frac{3}{2} + \frac{4}{3}e^t + \frac{1}{6}e^{-2t}.$$

c) Use Laplace transform to find the solution of

$$y'(t) - y(t) = e^t + e^{-t}, \qquad \text{with } y(0) = \pi.$$

• Applying the Laplace transform, we get

$$sY - \pi - Y = \frac{1}{s-1} + \frac{1}{s+1},$$

thus

$$Y = \frac{1}{(s-1)^2} + \frac{1}{s^2-1} + \frac{\pi}{(s-1)},$$

which gives

$$y(t) = \pi e^t + te^t + \sinh t = \pi e^t + te^t + \frac{e^t - e^{-t}}{2}.$$

2 **a)** Let $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ be the complex Fourier series of the following function

$$f(x) = 1 - x^2, \quad x \in (-\pi, \pi).$$

Compute c_n.

• Observe that $f(x)$ is an even function ($f(x) = f(-x)$). Its Fourier cosine series computes

$$f(x) = 1 - \frac{\pi^2}{3} + \sum_{n>0} \frac{4(-1)^{n+1}}{n^2} \cos(nx).$$

Now, using that $e^{inx} = \cos(nx) + i\sin(nx)$ and noticing that $f(x)\sin(nx)$ is an odd function, a direct computation gives

$$1 - x^2 = 1 - \frac{\pi^2}{3} - \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{2(-1)^n}{n^2} e^{inx}.$$

Thus

$$c_0 = 1 - \frac{\pi^2}{3},$$

and

$$c_n = \frac{2(-1)^{n+1}}{n^2}, \qquad n \neq 0.$$

b) Compute the Fourier transform of

$$f(x) = xe^{-|x|}.$$

• We can either compute it directly or use the identity $\frac{d}{d\omega}\hat{f}(\omega) = \mathcal{F}(-ixf(x))(\omega)$. In fact, we have

$$\begin{aligned}\mathcal{F}(e^{-|x|})(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-i\omega x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x} e^{-i\omega x} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{x} e^{-i\omega x} dx.\end{aligned}$$

After a few steps, this yields

$$\mathcal{F}(e^{-|x|})(\omega) = \sqrt{\frac{2}{\pi}} \frac{1}{\omega^2+1}.$$

Thus, using $\frac{d}{d\omega}\hat{f}(\omega) = \mathcal{F}(-ixf(x))(\omega)$, gives

$$\mathcal{F}(-ixe^{-|x|})(\omega) = \sqrt{\frac{2}{\pi}}\frac{-2\omega}{(\omega^2+1)^2},$$

which yields

$$\mathcal{F}(xe^{-|x|})(\omega) = \sqrt{\frac{2}{\pi}}\frac{-2i\omega}{(\omega^2+1)^2}.$$

3 **a) Mat 4N**: Show that for $a \neq 0$

$$\mathcal{F}(f(at))(\omega) = \frac{1}{|a|}\mathcal{F}(f(t))(\frac{\omega}{a})$$

- Recall that by definition we have

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\ \mathrm{d}x.$$

Thus with $a \neq 0$ we have

$$\mathcal{F}(f(at))(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(at)e^{-i\omega t}\ \mathrm{d}t.$$

Now replace at by x, we get $dx = adt$, or $t = x/a$ and $dt = \frac{1}{a}dx$. Then, for $a > 0$ we find

$$\mathcal{F}(f(at))(\omega) = \frac{1}{a}\mathcal{F}(f(t))(\frac{\omega}{a}).$$

For $a < 0$ we obtain

$$\mathcal{F}(f(at))(\omega) = -\frac{1}{a}\mathcal{F}(f(t))(\frac{\omega}{a}).$$

This implies that for $a \neq 0$

$$\mathcal{F}(f(at))(\omega) = \frac{1}{|a|}\mathcal{F}(f(t))(\frac{\omega}{a}).$$

b) Mat 4D: Show that the heat kernel $h(x,t) := \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$ satisfies $h_t = \frac{1}{2}h_{xx}$.

- If

$$h(x,t) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}},$$

then

$$h_t = \frac{1}{\sqrt{2\pi}}\cdot\frac{-1}{2}\cdot t^{\frac{-3}{2}}\cdot e^{-\frac{x^2}{2t}} + \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}}\cdot\frac{-x^2}{2}\cdot\frac{-1}{t^2} = \frac{x^2-t}{2t^2}h,$$

and

$$h_x = -\frac{x}{t}h,\ h_{xx} = -\frac{1}{t}h + \frac{x^2}{t^2}h = \frac{x^2-t}{t^2}h.$$

Thus we get that

$$h_t = \frac{1}{2}h_{xx}. \tag{1}$$

4 **a)** Solve the following heat equation

$$u_t = \frac{1}{2} u_{xx}, \quad t \geq 0, \quad 0 \leq x \leq \pi,$$

with the boundary conditions

$$u(t, 0) = u(t, \pi) = 0, \ \forall \ t \geq 0;$$

and the initial condition

$$u(0, x) = \sin 3x + \sin 5x, \ \forall \ 0 \leq x \leq \pi.$$

• **Step 1: Separating variables**: Find solutions of the form

$$u(t, x) = G(t)F(x).$$

Since

$$u_t = G'F, \quad u_{xx} = GF'',$$

our equation becomes

$$G'F = \frac{1}{2} GF'',$$

thus

$$\frac{2G'}{G} = \frac{F''}{F} \equiv k.$$

Step 2: Fit boundary conditions: Notice that the boundary conditions

$$G(t)F(0) = G(t)F(\pi) = 0,$$

are equivalent to

$$F(0) = F(\pi) = 0.$$

In case $k = 0$, then $F'' \equiv 0$, i.e $F(x) = ax + b$. The boundary conditions imply then that $F \equiv 0$.

In case $k = \mu^2 > 0$, then the general solution for

$$F'' = \mu^2 F$$

is $F(x) = Ae^{\mu x} + Be^{-\mu x}$. The boundary conditions imply this time that

$$A + B = 0, \quad Ae^{\mu\pi} + Be^{-\mu\pi} = 0,$$

thus $A = B = 0$.

Hence, the only possible case is $k = -p^2 < 0$, then the general solution for

$$F'' = -p^2 F$$

is $F(x) = A \cos px + B \sin px$, such that $F(0) = 0$ gives $A = 0$. Thus $F(x) = B \sin px$, $B \neq 0$, but $F(\pi) = 0$ gives $\sin p\pi = 0$, i.e.

$$p = n, \quad n = 1, 2, \ldots$$

(notice that $\sin -px = -\sin px$, thus up to a constant they give the same solution).

Summary: The boundary condition implies that $p^2 = n^2$, $n = 1, 2, \dots$, and

$$F(x) = F_n(x) = \sin nx.$$

Now we can solve

$$G' = -\frac{n^2}{2} G$$

and get

$$G_n(t) = B_n e^{-\frac{n^2}{2} t}.$$

Thus the general solution is

$$u(t, x) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2}{2} t} \sin nx.$$

Step 3: Fit the initial conditions: We have

$$u(0, x) = \sum_{n=1}^{\infty} B_n \sin nx = f(x).$$

By the uniqueness of the Fourier sine series, we get

$$B_3 = B_5 = 1, \quad B_n = 0, \quad \text{if } n \neq 3, 5.$$

Thus

$$u(t, x) = e^{-\frac{9}{2} t} \sin 3x + e^{-\frac{25}{2} t} \sin 5x.$$

5 **a)** Find a polynomial $p(x) \in \mathbb{P}_3$ interpolating the points

x_i	-2	0	1	2
y_i	0	1	$9/8$	0

• In this case, the easy solution is to use finite differences and Newton interpolation. The table of finite differences is:

$$\begin{array}{r|lllll}
-2 & 0 & & & \\
 & & 1/2 & & \\
0 & 1 & & -1/8 & \\
 & & 1/8 & & -1/8 \\
1 & 9/8 & & -5/8 & \\
 & & -9/8 & & \\
2 & 0 & & &
\end{array}$$

and the interpolation polynomial becomes:

$$p(x) = \frac{1}{2}(x+2) - \frac{1}{8}(x+2)x - \frac{1}{8}(x+2)x(x-1) = 1 + \frac{1}{2}x - \frac{1}{4}x^2 - \frac{1}{8}x^3.$$

6 The integral

$$\int_a^b f(x)dx$$

can be approximated by the quadrature formula

$$Q(a,b) = \frac{3h}{2}\Big(f(x_1) + f(x_2)\Big)$$

with

$$h = \frac{b-a}{3}, \quad x_1 = a + h \quad \text{and} \quad x_2 = a + 2h.$$

a) Apply the quadrature rule to the integral

$$\int_1^2 x\ln(x)dx.$$

• In this case, $h = 1/3$, $x_1 = 4/3$ and $x_2 = 5/3$, so the quadrature rule gives

$$Q(1,2) = \frac{1}{2}\left(\frac{4}{3}\ln(\frac{4}{3}) + \frac{5}{3}\ln(\frac{5}{3})\right) = 0.617476.$$

(For comparision, the exact integral $I(1,2) = 2\ln(2) - 3/4 = 0.636294$.)

b) Find the degree of precision of the quadrature rule. You can use the interval $[a,b] = [-1,1]$.

• The quadrature is of precision d if $\int_a^b x^k dx = Q(a,b)[x^k]$ for $k = 0, 1, \ldots, d$. And the choice of interval does not matter, so we use the suggested $[-1,1]$. In this case $h = 2/3, x_1 = -1/3$ and $x_2 = 1/3$, resulting in $Q(-1,1)[x^k] = x_1^k + x_2^k$.

k	$\int_{-1}^1 x^k dx$	$Q(-1,1)[x^k]$)
0	2	2
1	0	0
2	2/3	2/9

So, the precision of the method is only $d = 1$.

7 **a)** The following pyhton-code is given:

```
x = 2.5
for k in range(100):
    x_new = (3*x**4 + 24*x**2 -16)/(8*x**3)
    # Stop the iterations when ..
    x = x_new
```

Write down the fixed point iteration scheme which is implemented here.

Suggest an appropriate stopping criterium, and write down the corresponding python code.

- The fixed point iteration is $x_k = g(x_k)$ with

$$x_{k+1} = \frac{3x_k^4 + 24x_k^2 - 16}{8x_k^3}$$

and $x_0 = 2.5$.

Stop the iterations when $|x_{k+1} - x_k| \leq \text{Tol}$, where Tol is some user defined tolerance. So the code with a stopping criterium could be something like

```
x = 2.5
Tol = 1.e-4
for k in range(100):
    x_new = (3*x**4 + 24*x**2 -16)/(8*x**3)
    if abs(x_new-x) <= Tol:
        x = x_new
        break
    x = x_new
```

b) Given that the fixed point r is known, and all computations are done with very high accuracy. In this case, the error $e_k = |r - x_k|$ for each k would be printed out as follows:

```
k =  1,  error = 9.50e-03
k =  2,  error = 1.06e-07
k =  3,  error = 1.49e-22
k =  4,  error = 4.14e-67
```

Use this to estimate the rate of convergence for this iteration scheme.

- The rate of convergence is p if $e_{k+1} \approx Ce_k^p$. This can be estimated by

$$\begin{array}{c} e_{k+1} \approx Ce_k^p \\ e_{k+2} \approx Ce_{k+1}^p \end{array} \quad \Rightarrow \quad \frac{e_{k+1}}{e_{k+2}} \approx \left(\frac{e_k}{e_{k+1}}\right)^p \quad \Rightarrow \quad p \approx \frac{\log\left(\frac{e_{k+1}}{e_{k+2}}\right)}{\log\left(\frac{e_k}{e_{k+1}}\right)}$$

which for the results in the table gives:

$$k = 1 \qquad p \approx \frac{\log\left(\frac{1.06\cdot 10^{-7}}{1.49\cdot 10^{-22}}\right)}{\log\left(\frac{9.50\cdot 10^{-3}}{1.06\cdot 10^{-7}}\right)} = 3.00$$

$$k = 2 \qquad p \approx \frac{\log\left(\frac{1.49\cdot 10^{-22}}{4.14\cdot 10^{-67}}\right)}{\log\left(\frac{1.06\cdot 10^{-7}}{1.49\cdot 10^{-22}}\right)} = 3.00$$

Alternatively, just test for which p the factor e_{k+1}/e_k^p is almost constant:

k	e_{k+1}/e_k	e_{k+1}/e_k^2	e_{k+1}/e_k^3	e_{k+1}/e_k^4
1	$1.12 \cdot 10^{-5}$	$1.17 \cdot 10^{-3}$	0.124	13.1
2	$1.41 \cdot 10^{-15}$	$1.33 \cdot 10^{-8}$	0.125	$1.18 \cdot 10^{6}$
3	$2.78 \cdot 10^{-45}$	$1.86 \cdot 10^{-23}$	0.124	$8.39 \cdot 10^{20}$

So we observe cubic convergence ($p = 3$).

8 The following Runge–Kutta method is given:

$$\begin{aligned}\mathbf{k}_1 &= \mathbf{f}(x_n, \mathbf{y}_n),\\ \mathbf{k}_2 &= \mathbf{f}(x_n + \frac{h}{2}, \mathbf{y}_n + \frac{h}{2}\mathbf{k}_1),\\ \mathbf{y}_{n+1} &= \mathbf{y}_n + h\mathbf{k}_2.\end{aligned}$$

a) Do one step with step size $h = 0.1$ using the above method on the problem:

$$\begin{aligned} y_1' &= y_1 + xy_2^2, \qquad & y_1(1) &= 1.0,\\ y_2' &= y_1 y_2, & y_2(1) &= -1.0.\end{aligned}$$

- We have:

$$\mathbf{y} = [y_1, y_2]^T, \qquad \mathbf{f} = [y_1 + xy_2^2,\ y_1 y_2]^T,$$

and the initial values are:

$$\mathbf{y}(1) = \mathbf{y}_0 = [1, -1]^T, \quad x_0 = 1.$$

So we get for $n = 0$:

$$\begin{aligned}\mathbf{k}_1 &= [2, -1]^T,\\ \mathbf{y}_0 + \frac{h}{2}\mathbf{k}_1 &= [1.10, -1.05]^T\\ \mathbf{k}_2 &= [2.2576, -1.155]^T\\ \mathbf{y}_1 &= [1.2258, -1.116]^T.\end{aligned}$$

b) Find the stability function $R(z)$ for this function. Find also the corresponding stability interval.

- Use the method to solve the linear test equation

$$y' = \lambda y, \qquad \lambda < 0.$$

The stability function $R(z)$ is defined by

$$y_{n+1} = R(z) y_n, \qquad z = h\lambda,$$

which in this case can be found by:

$$\begin{aligned} k_1 &= \lambda y_n\\ k_2 &= \lambda(y_n + \frac{h}{2}\lambda y_n) = \lambda\left(1 + \frac{h\lambda}{2}\right)y_n\\ y_{n+1} &= y_n + h\lambda\left(1 + \frac{h\lambda}{2}\right)y_n = (1 + z + \frac{z^2}{2})y_n\end{aligned}$$

so the stability function is

$$R(z) = 1 + z + \frac{z^2}{2}.$$

The stability interval is defined

$$\mathcal{S} = \{z \in \mathbb{R} \ : \ |R(z)| \le 1\},$$

that is the set of z for which both inequalities

$$1 + z + \frac{z^2}{2} \le 1 \qquad \text{and} \qquad 1 + z + \frac{z^2}{2} \ge -1$$

are satisfied. The first is satisfied if $-2 \le z \le 0$, the second is satisfied for all z. So we can conclude that

$$\mathcal{S} = [-2, 0].$$

9 **a)** In this exercise you are asked to set up a finite difference scheme for the two point boundary value problem

$$u'' + 2u = x^2, \qquad u'(0) + u(0) = 0, \qquad u(1) = 2,$$

defined on the interval $0 \le x \le 1$.
Let N be the number of grid points with $h = 1/N$, and let U_i be the approximations to the exact solution $u(x_i)$ in the gridpoints $x_i = ih$ for $i = 0, 1, \dots, N$. Set up the finite difference scheme for a general N in the form

$$A\mathbf{U} = \mathbf{b},$$

where $\mathbf{U} = [U_0, U_1, \dots, U_N]^T$, that is, set up the matrix A and the vector $\mathbf{b}$.

- Approximate $u''(x)$ at some grid point x_i by a central difference formula:

$$u''(x_i) \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2},$$

and let $U_i \approx u(x_i)$ be the approximation to the solution in the gridpoints. So, for each inner gridpoint, the difference formula corresponding to the differential equation is

$$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + 2U_i = x_i^2, \qquad i = 1, 2, \dots, N-1. \tag{*}$$

From the right boundary condition we see that $U_N = u(1) = 2$. The left boundary condition is treated by using a false boundary, assuming we have an artifical grid point $x_{-1} = -h$. Then, using a central difference

$$u'(0) \approx \frac{u(x_1) - u(x_{-1})}{2h}$$

and let $U_{-1} \approx u(x_{-1})$ the discrete version of the boundary condition $u'(0) + u(0) = 0$ is

$$\frac{U_1 - U_{-1}}{2h} + U_0 = 0.$$

So there are two difference equations for $i = 0$, that is, this one describing the boundary condition and (*) approximating the equation. Solve the boundary difference equation with respect to U_{-1}:

$$U_{-1} = U_1 + 2hU_0.$$

Use this in (*) with $i = 0$:

$$\frac{U_1 - 2U_0 + (U_1 + 2hU_0)}{h^2} + 2U_0 = 0^2 \qquad \Rightarrow \qquad \frac{2U_1 - (2 - 2h)U_0}{h^2} + 2U_0 = 0.$$

Multiplying by h^2 on both sides, and using $x_i = ih$ gives the following scheme:

$$\begin{aligned} -2(1 - h - h^2)U_0 + 2U_1 &= 0, \\ U_{i-1} - 2(1 - h^2)U_i + U_{i+1} &= h^2 x_i^2, \qquad\qquad i = 1, 2, \dots, N-1. \\ U_N &= 2. \end{aligned}$$

This can be written as a linear system of equations $A\mathbf{U} = \mathbf{b}$, that is

$$\begin{bmatrix} -2(1-h-h^2) & 2 & 0 & \cdots & 0 & 0 \\ 1 & -2(1-h^2) & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2(1-h^2) & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & 1 & -2(1-h^2) & 1 & 0 \\ 0 & & & 1 & -2(1-h^2) & 1 \\ 0 & \cdots & & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ \vdots \\ \vdots \\ U_{N-2} \\ U_{N-1} \\ U_N \end{bmatrix} = \begin{bmatrix} 0 \\ h^2 x_1^2 \\ h^2 x_2^2 \\ \vdots \\ \vdots \\ h^2 x_{N-2}^2 \\ h^2 x_{N-1}^2 \\ 2 \end{bmatrix}.$$