



Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4130/35 Mathematics 4N/4D**

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Examination date: December 14 2018

Examination time (from–to): 09:00–13:00

Permitted examination support material: Code C: Approved calculator

One yellow, stamped A5 sheet with own handwritten formulas and notes (on both sides)

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- There are two versions of Problem 3: one for Mathematics 4N and one for Mathematics 4D.
- Good Luck!

Language: English

Number of pages: 4

Number of pages enclosed: 1

Checked by:

Informasjon om trykking av eksamensoppgave

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Problem 1 Laplace Transform [20 points]

- a) Compute Laplace transform of

$$f(t) = te^t.$$

- b) Compute the inverse Laplace transform
- $\mathcal{L}^{-1}(F)(t)$
- of the following function

$$F(s) := \frac{s+3}{s(s-1)(s+2)}.$$

(Hint: you can use partial fraction decomposition).

- c) Use Laplace transform to find the solution of

$$y'(t) - y(t) = e^t + e^{-t}, \quad \text{with } y(0) = \pi.$$

Problem 2 Fourier Series and Transform [14 points]

- a) Let
- $\sum_{n \in \mathbb{Z}} c_n e^{inx}$
- be the complex Fourier series of the following function

$$f(x) = 1 - x^2, \quad x \in (-\pi, \pi).$$

Compute c_n .

- b) Compute the Fourier transform of

$$f(x) = xe^{-|x|}.$$

Problem 3 TMA4130 Mathematics 4N: Fourier Transform [6 points]

Show that for $a \neq 0$

$$\mathcal{F}(f(at))(\omega) = \frac{1}{|a|} \mathcal{F}(f(t))\left(\frac{\omega}{a}\right)$$

Problem 3 TMA4135 Mathematics 4D: Partial Derivative [6 points]

Show that the heat kernel $h(x, t) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ satisfies $h_t = \frac{1}{2} h_{xx}$.

Problem 4 Partial Differential Equations [10 points]

Solve the following heat equation

$$u_t = \frac{1}{2}u_{xx}, \quad t \geq 0, \quad 0 \leq x \leq \pi,$$

with the boundary conditions

$$u(t, 0) = u(t, \pi) = 0, \quad \forall t \geq 0;$$

and the initial condition

$$u(0, x) = \sin 3x + \sin 5x, \quad \forall 0 \leq x \leq \pi.$$

Problem 5 Polynomial Interpolation [10 points]

Find a polynomial $p(x) \in \mathbb{P}_2$ interpolating the points

$$\begin{array}{c|cccc} x_i & -2 & 0 & 1 & 2 \\ \hline y_i & 0 & 1 & 9/8 & 0 \end{array}.$$

Problem 6 Numerical Integration [10 points]

The integral

$$\int_a^b f(x)dx,$$

can be approximated by the quadrature formula

$$Q(a, b) = \frac{3h}{2} \left(f(x_1) + f(x_2) \right),$$

with

$$h = \frac{b-a}{3}, \quad x_1 = a + h \quad \text{and} \quad x_2 = a + 2h.$$

a) Apply the quadrature rule to the integral

$$\int_1^2 x \ln(x) dx.$$

b) Find the degree of precision of the quadrature rule. You can use the interval $[a, b] = [-1, 1]$.

Problem 7 Numerical Solution of Nonlinear Equations [10 points]

a) The following python-code is given:

```
x = 2.5
for k in range(100):
    x_new = (3*x**4 + 24*x**2 - 16)/(8*x**3)
    # Stop the iterations when ..
    x = x_new
```

Write down the fixed point iteration scheme which is implemented here.

Suggest an appropriate stopping criterium, and write down the corresponding python code.

b) Given that the fixed point r is known, and all computations are done with very high accuracy. In this case, the error $e_k = |r - x_k|$ for each k would be printed out as follows:

```
k = 1, error = 9.50e-03
k = 2, error = 1.06e-07
k = 3, error = 1.49e-22
k = 4, error = 4.14e-67
```

Use this to estimate the rate of convergence for this iteration scheme.

Problem 8 Ordinary Differential Equations [10 points]

The following Runge–Kutta method is given:

$$\begin{aligned}\mathbf{k}_1 &= \mathbf{f}(x_n, \mathbf{y}_n), \\ \mathbf{k}_2 &= \mathbf{f}\left(x_n + \frac{h}{2}, \mathbf{y}_n + \frac{h}{2}\mathbf{k}_1\right), \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + h\mathbf{k}_2.\end{aligned}$$

a) Do one step with step size $h = 0.1$ using the above method on the problem:

$$\begin{aligned}y_1' &= y_1 + xy_2^2, & y_1(1) &= 1.0, \\ y_2' &= y_1y_2, & y_2(1) &= -1.0.\end{aligned}$$

b) Find the stability function $R(z)$ for this ~~function~~ **method**. Find also the corresponding stability interval.

Problem 9 Finite difference scheme [10 points]

In this exercise you are asked to set up a finite difference scheme for the two point boundary value problem

$$u'' + 2u = x^2, \quad u'(0) + u(0) = 0, \quad u(1) = 2,$$

defined on the interval $0 \leq x \leq 1$.

Let N be the number of grid points with $h = 1/N$, and let U_i be the approximations to the exact solution $u(x_i)$ in the gridpoints $x_i = ih$ for $i = 0, 1, \dots, N$. Set up the finite difference scheme for a general N in the form

$$A\mathbf{U} = \mathbf{b},$$

where $\mathbf{U} = [U_0, U_1, \dots, U_N]^T$, that is, set up the matrix A and the vector \mathbf{b} .

Fourier Transform

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$	$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-w^2/4a}$
$e^{-a x }$	$\sqrt{\frac{2}{\pi}} \frac{a}{w^2 + a^2}$
$\frac{1}{x^2 + a^2}$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a w }}{a}$
$\begin{cases} 1 & \text{for } x < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin wa}{w}$

Laplace Transform

$f(t)$	$F(s) = \int_0^{\infty} e^{-st} f(t) dt$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}},$ <small>for $n = 0, 1, 2, \dots, \Gamma(n+1) = n!$</small>
e^{at}	$\frac{1}{s - a}$
$\delta(t - a)$	e^{-as}

$$\int x^n \cos ax \, dx = \frac{1}{a} x^n \sin ax - \frac{n}{a} \int x^{n-1} \sin ax \, dx$$

$$\int x^n \sin ax \, dx = -\frac{1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax \, dx$$