NTNU
Norwegian University of
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Department of Mathematical Sciences

Examination paper for **TMA4130/35 Mathematics 4N/4D**

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Examination date: December 14 2018

Examination time (from–to): 09:00–13:00

Permitted examination support material: Code C: Approved calculator
One yellow, stamped A5 sheet with own handwritten formulas and notes (on both sides)

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- There are two versions of Problem 3: one for Mathematics 4N and one for Mathematics 4D.
- Good Luck!

Language: English

Number of pages: 4

Number of pages enclosed: 1

Checked by:

Informasjon om trykking av eksamensoppgave	
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Date Signature

Problem 1 <u>Laplace Transform</u> [20 points]

a) Compute Laplace transform of

$$f(t) = te^t.$$

b) Compute the inverse Laplace transform $\mathcal{L}^{-1}(F)(t)$ of the following function

$$F(s) := \frac{s+3}{s(s-1)(s+2)}.$$

(Hint: you can use partial fraction decomposition).

c) Use Laplace transform to find the solution of

$$y'(t) - y(t) = e^t + e^{-t}, \qquad \text{with } y(0) = \pi.$$

Problem 2 <u>Fourier Series and Transform</u> [14 points]

a) Let $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ be the complex Fourier series of the following function

$$f(x) = 1 - x^2, \quad x \in (-\pi, \pi).$$

Compute c_n.

b) Compute the Fourier transform of

$$f(x) = xe^{-|x|}.$$

Problem 3 *TMA4130 Mathematics 4N:* <u>Fourier Transform</u> [6 points]

Show that for $a \neq 0$

$$\mathcal{F}(f(at))(\omega) = \frac{1}{|a|}\mathcal{F}(f(t))(\frac{\omega}{a})$$

Problem 3 *TMA4135 Mathematics 4D:* <u>Partial Derivative</u> [6 points]

Show that the heat kernel $h(x,t) := \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$ satisfies $h_t = \frac{1}{2}h_{xx}$.

Problem 4 Partial Differential Equations [10 points]

Solve the following heat equation

$$u_t = \frac{1}{2} u_{xx}, \quad t \geq 0, \quad 0 \leq x \leq \pi,$$

with the boundary conditions

$$u(t, 0) = u(t, \pi) = 0, \ \forall \ t \geq 0;$$

and the initial condition

$$u(0, x) = \sin 3x + \sin 5x, \ \forall \ 0 \leq x \leq \pi.$$

Problem 5 Polynomial Interpolation [10 points]

Find a polynomial $p(x) \in \mathbb{P}_2$ interpolating the points

x_i	-2	0	1	2
y_i	0	1	$9/8$	0

Problem 6 Numerical Integration [10 points]

The integral

$$\int_a^b f(x) dx,$$

can be approximated by the quadrature formula

$$Q(a, b) = \frac{3h}{2} \Big(f(x_1) + f(x_2) \Big),$$

with

$$h = \frac{b-a}{3}, \quad x_1 = a + h \quad \text{and} \quad x_2 = a + 2h.$$

a) Apply the quadrature rule to the integral

$$\int_1^2 x \ln(x) dx.$$

b) Find the degree of precision of the quadrature rule. You can use the interval $[a, b] = [-1, 1]$.

Problem 7 Numerical Solution of Nonlinear Equations [10 points]

a) The following pyhton-code is given:

```
x = 2.5
for k in range(100):
    x_new = (3*x**4 + 24*x**2 -16)/(8*x**3)
    # Stop the iterations when ..
    x = x_new
```

Write down the fixed point iteration scheme which is implemented here.

Suggest an appropriate stopping criterium, and write down the corresponding python code.

b) Given that the fixed point r is known, and all computations are done with very high accuracy. In this case, the error $e_k = |r - x_k|$ for each k would be printed out as follows:

```
k =  1,  error = 9.50e-03
k =  2,  error = 1.06e-07
k =  3,  error = 1.49e-22
k =  4,  error = 4.14e-67
```

Use this to estimate the rate of convergence for this iteration scheme.

Problem 8 Ordinary Differential Equations [10 points]

The following Runge–Kutta method is given:

$$
\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(x_n, \mathbf{y}_n), \\
\mathbf{k}_2 &= \mathbf{f}(x_n + \frac{h}{2}, \mathbf{y}_n + \frac{h}{2}\mathbf{k}_1), \\
\mathbf{y}_{n+1} &= \mathbf{y}_n + h\mathbf{k}_2.
\end{aligned}
$$

a) Do one step with step size $h = 0.1$ using the above method on the problem:

$$
\begin{aligned}
y_1' &= y_1 + x y_2^2, & y_1(1) &= 1.0, \\
y_2' &= y_1 y_2, & y_2(1) &= -1.0.
\end{aligned}
$$

b) Find the stability function $R(z)$ for this function. Find also the corresponding stability interval.

Problem 9 Finite difference scheme [10 points]

In this exercise you are asked to set up a finite difference scheme for the two point boundary value problem

$$u'' + 2u = x^2, \qquad u'(0) + u(0) = 0, \qquad u(1) = 2,$$

defined on the interval $0 \leq x \leq 1$.

Let N be the number of grid points with $h = 1/N$, and let U_i be the approximations to the exact solution $u(x_i)$ in the gridpoints $x_i = ih$ for $i = 0, 1, \ldots, N$. Set up the finite difference scheme for a general N in the form

$$A\mathbf{U} = \mathbf{b},$$

where $\mathbf{U} = [U_0, U_1, \ldots, U_N]^T$, that is, set up the matrix A and the vector $\mathbf{b}$.

Fourier Transform

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}\ \mathrm{d}w$	$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}\ \mathrm{d}x$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-w^2/4a}$
$e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}} \frac{a}{w^2+a^2}$
$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\|w\|}}{a}$
$\begin{cases} 1 & \text{for } \|x\| < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin wa}{w}$

Laplace Transform

$f(t)$	$F(s) = \int_0^{\infty} e^{-st} f(t)\ \mathrm{d}t$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$, for $n = 0, 1, 2, \ldots$, $\Gamma(n+1) = n!$
e^{at}	$\frac{1}{s-a}$
$\delta(t-a)$	e^{-as}

$$\int x^n \cos ax\ \mathrm{d}x = \frac{1}{a} x^n \sin ax - \frac{n}{a} \int x^{n-1} \sin ax\ \mathrm{d}x$$
$$\int x^n \sin ax\ \mathrm{d}x = -\frac{1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax\ \mathrm{d}x$$