Numerics for PDEs (21.4, 21.6)

We find numerical methods for quasilinear PDEs of order 2

$$a u_{xx} + 2b u_{xy} + c u_{yy} = F(x, y, u_x, u_y)$$

$u(x,y)$, $a, b, c \in \mathbb{R}$

<u>Definition</u> The PDE is of type

a) <u>Elliptic</u> if $ac - b^2 > 0$

e.g. $u_{xx} + u_{yy} = f(x,y)$

(Poisson equation)

$a = 1, b = 0, c = 1$

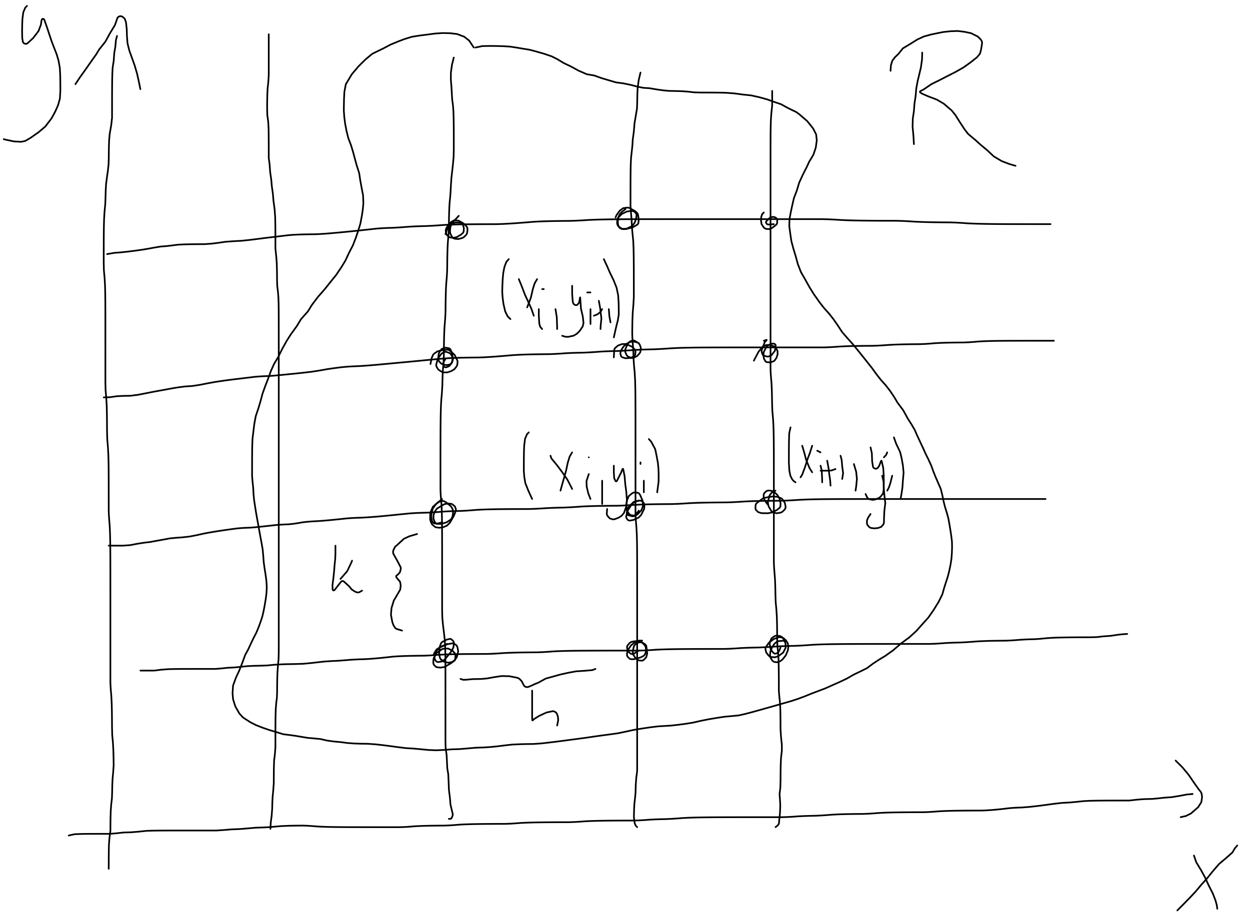
b) <u>Parabolic</u> if $ac - b^2 = 0$

e.g. $u_{xx} = u_y$ $a=1, b=0, c=0$
(Heat equation)

c) <u>Hyperbolic</u> if $ac - b^2 < 0$

e.g. $u_{xx} = u_{yy}$ (Wave equation)

$a = 1, \ b = 0, \ c = -1$

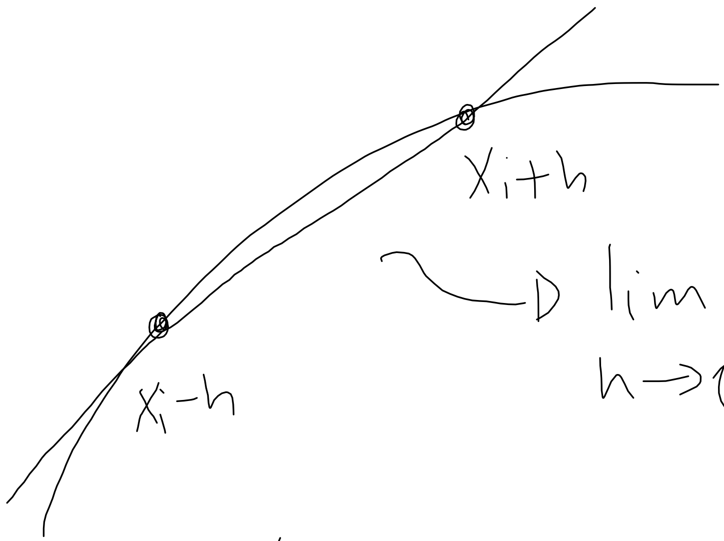


We will approximate the solution $u(x,y)$ at the <u>mesh points</u> in the region R.

<u>Difference equations</u>

$$x_{i+1} = x_i + h, \quad y_{i+1} = y_i + k$$

$$u_x(x_i, y_j) = \lim_{h \to 0} \frac{u(x_i+h, y_j) - u(x_i-h, y_j)}{2h}$$



$\rightarrow \lim_{h\to 0}$ slope secant line at x_i+h & x_i-h.

$$\approx \frac{u(\overset{x_{i+1}}{\overset{\parallel}{x_i+h}}, y_j) - u(\overset{x_{i-1}}{\overset{\parallel}{x_i-h}}, y_j)}{2h}$$

(h small)

$$= \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

Similarly,

$$u_y(x_i, y_j) = \lim_{k \to 0} \frac{u(x_i, y_j + k) - u(x_i, y_j - k)}{2k}$$

$$\approx \frac{u_{i,j+1} - u_{i,j-1}}{2K}$$

We apply this to get an approximation of the second derivative:

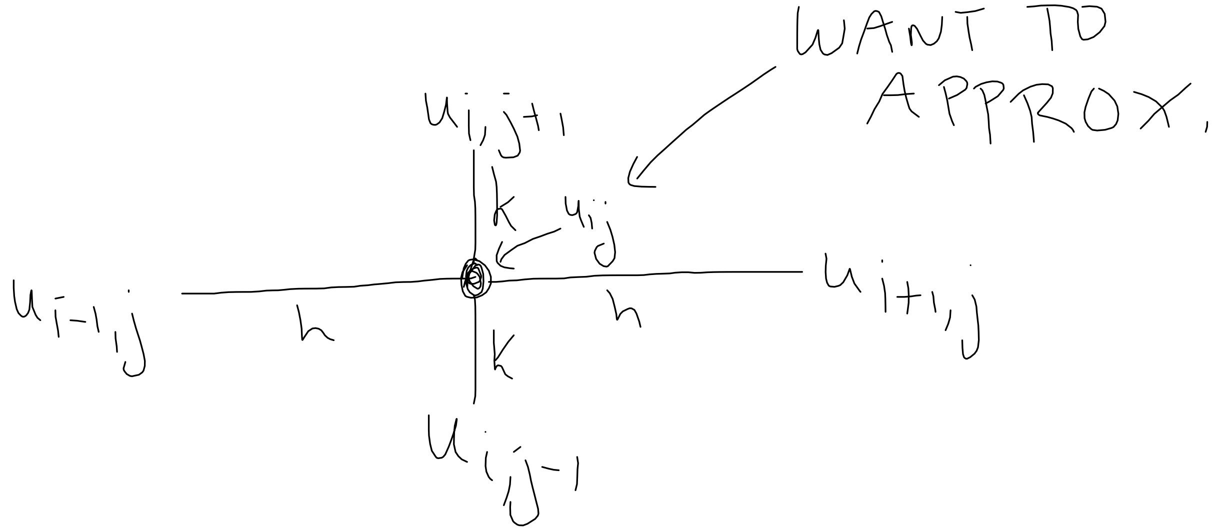
$$u_{xx}(x_i, y_j) = \lim_{h \to 0} \frac{u_x(x_i + h/2, y_j) - u_x(x_i - \frac{h}{2}, y_j)}{h}$$

$$\approx \frac{u_x(x_i+\frac{h}{2},y_j) - u_x(x_i-\frac{h}{2},y_j)}{h}$$

$$\approx \frac{\left(\frac{u(x_i+h/2+h/2,y_j) - u(x_i+h/2-h/2,y_j)}{h} - \frac{u(x_i-h/2+h/2,y_j) - u(x_i-h/2-h/2,y_j)}{h} \right)}{h}$$

$$= \boxed{\frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2}}$$

$$\boxed{u_{yy}(x_i, y_j) \approx \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{k^2}}$$



Laplace and Poisson equations

Laplace:

$$\nabla^2 u = \boxed{u_{xx} + u_{yy} = 0}$$

Poisson: $\boxed{u_{xx} + u_{yy} = f(x,y)}$

Apply differences equation:

$h = k$

$$\underbrace{\frac{u_{i+1,j} + u_{i-1,j} - 2u_{ij}}{h^2}}_{u_{xx}} + \underbrace{\frac{u_{i,j+1} + u_{i,j-1} - 2u_{ij}}{h^2}}_{u_{yy}} = f(x,y)$$

$$\boxed{u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} \approx h^2 f(x_i, y_j)}$$

$\iff$

$$u(x_i+h, y_j) + u(x_i, y_j+h) + u(x_i-h, y_j) + u(x_i, y_j-h) - 4u(x_i, y_j) \approx h^2 f(x_i, y_j)$$

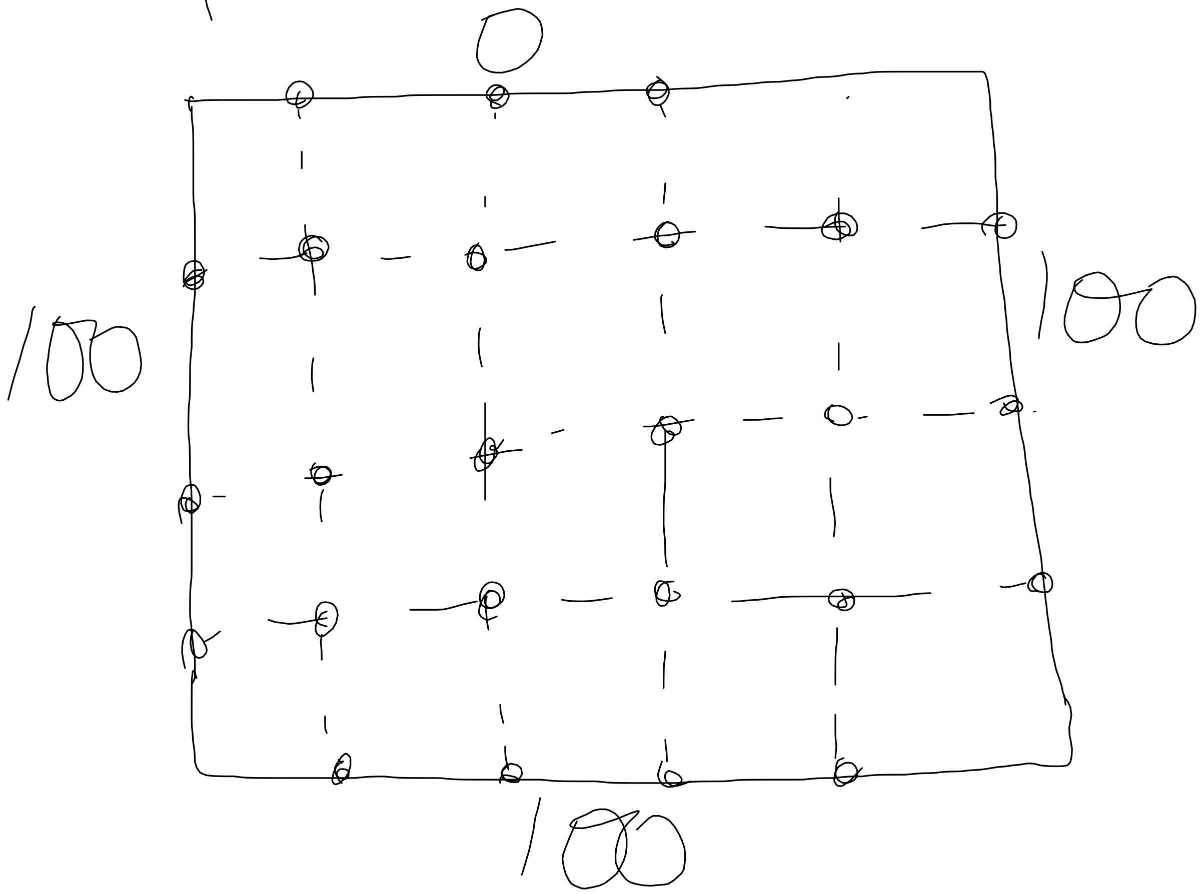
Dirichlet Problem

A Dirichlet problem is a PDE where u is prescribed on the boundary curve C of R.

Example: The four sides of a square plate of side 12 cm are kept at constant temperature, with

$$u(x,12)=0,\quad u(0,y)=u(12,y)$$
$$=u(x,0)=100$$

Where $u(x,y)$ is the temperature. Approximate the steady-state temperature at the mesh points



Heat equation

$$u_t = c^2(u_{xx} + u_{yy})$$

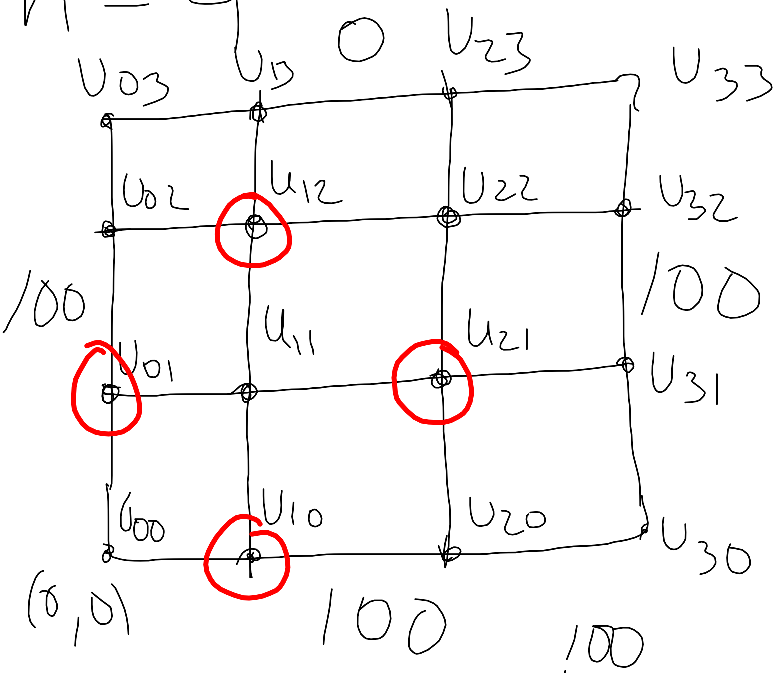
0 (indep. of time = steady-state)

$$\leadsto u_{xx} + u_{yy} = 0$$

(Laplace equation)

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} \approx 0$$

$h = 4$



1) $u_{21} + u_{12} + \overset{=100}{u_{01}} + \overset{=100}{u_{10}} - 4u_{11} = 0$

2) $\overset{=100}{u_{31}} + u_{22} + u_{11} + \overset{=100}{u_{20}} - 4u_{21} = 0$

3) $\overset{=100}{u_{32}} + \overset{=0}{u_{23}} + u_{12} + u_{21} - 4u_{22} = 0$

4) $u_{22} + \overset{=0}{u_{13}} + \underset{=100}{u_{02}} + u_{11} - 4u_{12} = 0$

1) $-4u_{11} + u_{21} + u_{12} = -200$

2) $u_{11} - 4u_{21} + u_{22} = -200$

3) $u_{11} - 4u_{12} + u_{22} = -100$

4) $u_{21} + u_{12} - 4u_{22} = -100$

Solve this system
(by iteration methods or directly)

$$\boxed{\begin{aligned} u_{11} &= u_{21} = 87.5 \\ u_{12} &= u_{22} = 62.5 \end{aligned}}$$

Parabolic equations

In this case, one needs additional conditions to ensure converge when $h \longrightarrow 0$.

1D Heat equation

$$(*) \quad u_t = c^2 u_{xx} \qquad u(x,t)$$

This models the temperature in a bar of length L $(0 \leq x \leq L)$ at time $t \geq 0$.

Initial conditions

$$u(x,0) = f(x)$$

Boundary conditions

$$u(0,t) = 0 \quad u(L,t) = 0$$

Applying the difference equations and replacing in (*) we have

$$u_t = \lim_{k \to 0} \frac{u(x,t+k) - u(x,t)}{k}$$

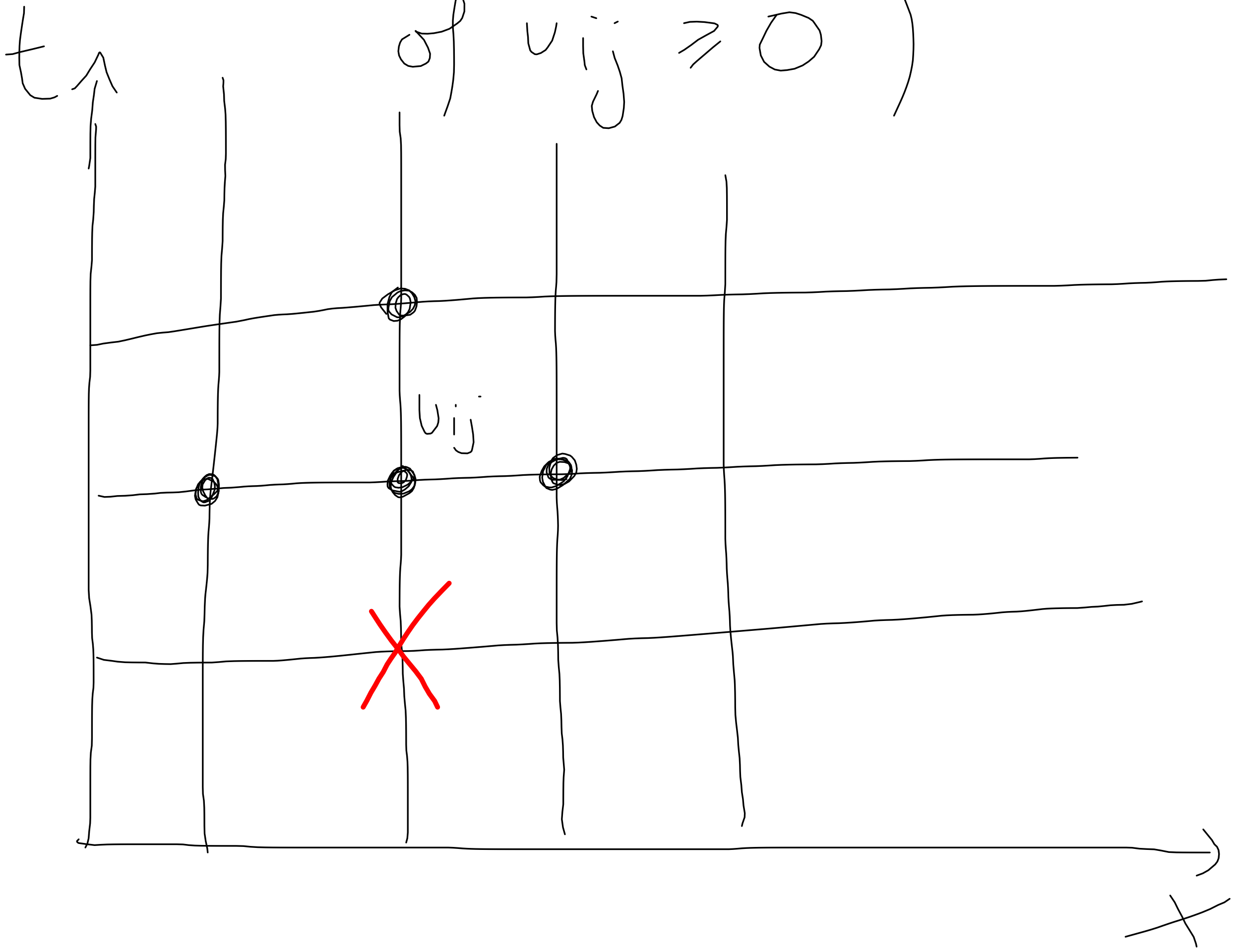
$$\approx \frac{u_{i,j+1} - u_{i,j}}{k} \quad \text{(forward difference)}$$

$$u_t = c^2 u_{xx}$$

$$\Rightarrow \boxed{\frac{u_{i,j+1} - u_{i,j}}{k} = c^2 \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}}$$

$$\Rightarrow \boxed{u_{i,j+1} = (1-2r)u_{ij} + r\left(u_{i+1,j} + u_{i-1,j}\right) \quad r = \frac{c^2 k}{h^2}}$$

The method converges
if $r \leq 1/2$.
($\Leftrightarrow$ coefficient in front
of $u_{ij} \geq 0$)



Example: Approximate the solution of the heat equation at $t=1$, $c=1$, $h=1$, $k=0.5$ in a bar of length 3 and initial temperature $f(x)=x(1-0.1x)=u(x,0)$

$$u_t = u_{xx}$$

In this case, we have

$$r = \frac{1}{2} = \frac{k}{h^2}$$