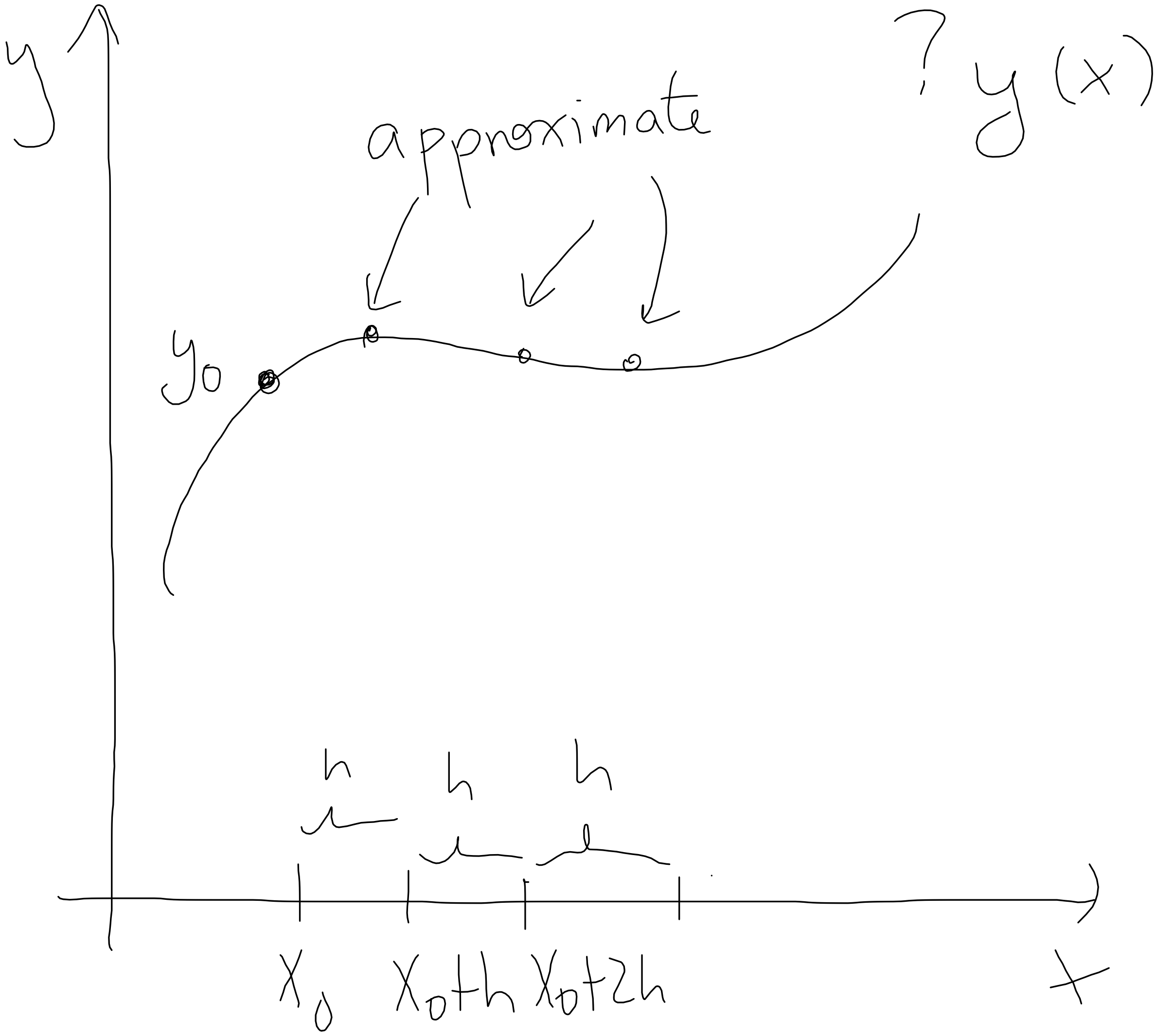
First Order ODEs (21.1)

Initial value problem:

$$y' = f(x, y) \quad , \quad y(x)$$

Ex.: $y' = \cos y + e^{-x}$

Initial Cond. $y(x_0) = y_0$



Choose h = step

The goal is to approximate numeric values of the solution $y(x)$ at equidistant points

$$x_1 = x_0 + h, \; x_2 = x_0 + 2h, \; \ldots$$

h (step size) is a fixed number.

Euler's method

Use Taylor first-order approximation

$y(x) = y(x_0) + y'(x_0)(x - x_0)$

$h = x - x_0$

$x_0 = x_n$

$$y(x_{n+1}) = y(x_n + h)$$

$$y_{n+1} \overset{\text{Taylor}}{\approx} y(x_n) + y'(x_n) \cdot h$$

$$\overset{\text{ODE}}{=} \boxed{y(x_n) + h\, f(x_n, y_n)}$$

Example: $y' = y + x$

$h = 0.1$

$y(0) = 1$, with $0 = x_0$ and $1 = y_0$

$$x_1 = x_0 + h$$

$$\underbrace{y(0.1)}_{y_1} \approx y(0) + 0.1(0+1)$$

$$y(x_0) + h\underbrace{f(x_0, y_0)}_{x_0 + y_0}$$

$$= 1 + 0.1(0+1) = 1.1$$

$$y(0.2) \simeq y_1 + h f(x_1, y_1)$$
$$= y_1 + h(x_1 + y_1)$$
$$= 1.1 + 0.1(0.1 + 1.1)$$
$$= 1.22$$

<u>Error of the Euler method</u>

Taylor's first approximation has remainder

$$y(x_n + h) = y(x_n) + h y'(x_n) + \underbrace{\frac{1}{2} h^2 y''(t)}_{\text{error}}$$

$t \in (x_n, x_n + h)$

This is the error at x_{n+1}.
However, Euler's method uses an approximation on $y(x_n)$ as well

$\ast$

Local error:

$$\varepsilon_L = \frac{1}{2} h^2 y''(t) = O(h^2)$$

order

$$0 < \lim_{h \to \infty} \frac{|\varepsilon_L|}{|h^2|} < \infty$$

Global error

If we want to approximate at equidistant points

$x_1, \dots, x_N$ in the interval (a,b)

$$\Rightarrow h = \frac{b-a}{N} \Rightarrow N = \frac{b-a}{h}$$

$$\varepsilon_G = \varepsilon_L \cdot N$$

$$= \frac{1}{2} h^{\cancel{2}} y''(t) \cdot \frac{b-a}{\cancel{h}}$$

$$= \frac{1}{2}(b-a)h\,y''(t) = O(h)$$

$\Rightarrow$ This is a <u>first-order method</u>.

$\mathcal{E}_G$ is given by an accumulation of $\mathcal{E}_L$ at each point.

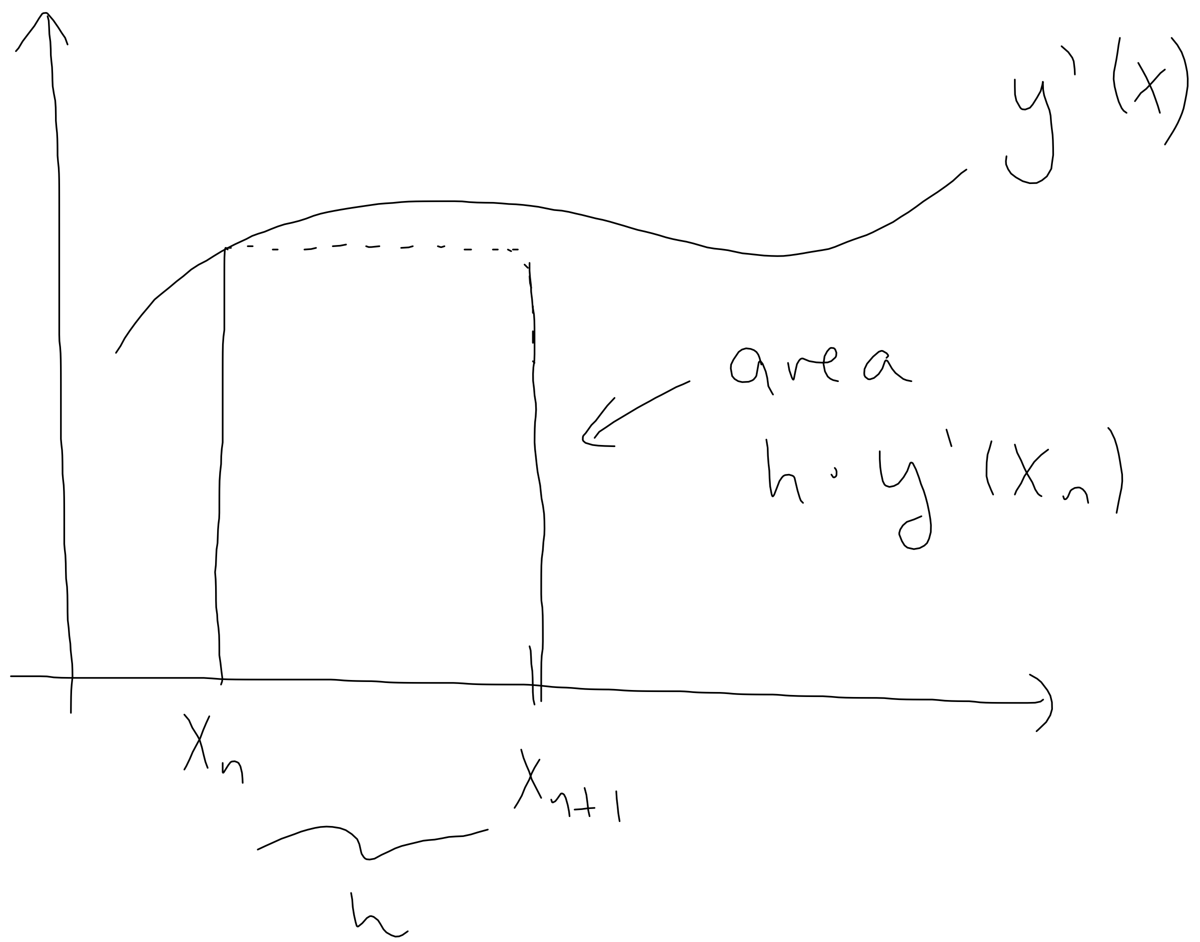
Improved Euler's method

We can view Euler's method as follows.

Fundamental thm of Calculus

$$y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} y'(t)\,dt$$

$$\Rightarrow y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} y'(t)\,dt$$



Euler's method: approximate the area by a rectangle

$\approx y_n + h\, y'(x_n)$

$$= y_n + h \cdot f(x_n, y_n).$$

<u>Improved Euler's method</u>

$\rightsquigarrow$ approximate the area by a trapezoid.

$$y_{n+1} \approx y_n + \frac{h}{2}\left[y'(x_n) + y'(x_{n+1})\right]$$

$$= y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

?

The strategy is to first approximate y_{n+1} by Euler's method, and then use it in the previous formula to get a better approximation.

predicator

1) $y^*_{n+1} = y_n + h f(x_n, y_n)$

2) $y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1}) \right]$

Corrector

Example: $y' = x + y$, $y(0) = 1$ } $h = 0.1$

1) $y_1^* = y^*(0.1) = 1.1$

(previous calculation)

2) $y_1 \approx y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^*)\right]$

$= 1 + 0.05\left[(0+1) + (0.1 + 1.1)\right]$

$= 1.11$

1) $y_2^* = y^*(0.2)$

$$= y_1 + hf(x_1, y_1)$$

$$= 1.11 + 0.1(0.1 + 1.11)$$

$$= 1.231$$

2) $y_2 \approx 1.11 + 0.05\left[(0.1 + 1.11) + (0.2 + 1.231)\right]$

$$= 1.242$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_{n+1}, y_n + k_1)$$

$$\Rightarrow \boxed{y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)}$$

Error: Local error is $O(h^3)$

Global error is $O(h^2)$

Runge-Kutta (RK) Method

$$y_{n+1} = y_n + \underbrace{\int_{x_n}^{x_{n+1}} y'(t)\,dt}_{}$$

We approximate by Simpson's rule on the points

x_n, $x_n + \frac{h}{2}$, x_{n+1}

(distance between each point is $h/2$)

$$y_{n+1} \approx y_n + \frac{h/2}{3}\Big[y'(x_n) + 4y'(x_n + h/2) + y'(x_{n+1})\Big]$$

$$= y_n + \frac{1}{6}\Big[hf(x_n, y_n) + 4hf(x_n + h/2, \underbrace{y(x_n + h/2)}_{?}) + hf(x_{n+1}, \underbrace{y_{n+1}}_{?})\Big]$$

We use predictors. We choose

1st term

$$k_1 = hf(x_n, y_n)$$

2nd term

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

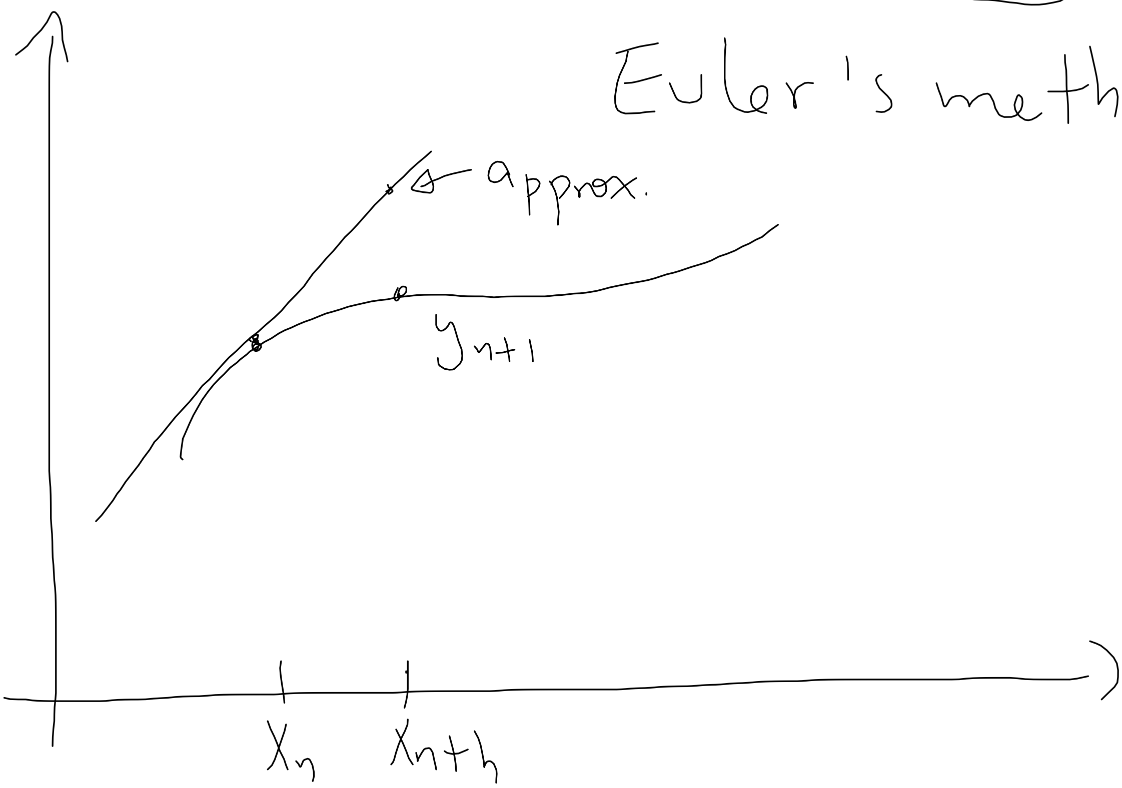
$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_{n+1}, y_n + k_3)$$

last term

$$y_{n+1} \approx y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Euler's method



<u>Example:</u> $y' = x + y$, $y(0) = 1$ $\quad h = 0.1$

$$k_1 = hf(x_0, y_0) = 0.1(0+1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1\left(0.05 + (1 + 0.05)\right) = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\left(0.05 + (1 + 0.055)\right)$$

$$= 0.1105$$

$$k_4 = hf(x_{n+1}, y_n + k_3)$$

$$= 0.1(0.1 + (1 + 0.1105))$$
$$= 0.12105$$

$$y_1 = y(0.1) \approx 1 + \frac{1}{6}(0.1 + 2 \cdot 0.11 + 2 \cdot 0.1105 + 0.12105)$$

$$\approx 1.1103$$

<u>Error</u>: The RK method has local error $O(h^5)$ and global error $O(h^4)$.

Backward Euler's method

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$$

Implicit equation

Must solve first or approximate a solution to the implicit equation by Newton's method or fixed-point method.

Error: Local $\mathcal{O}(h^2)$
global $\mathcal{O}(h)$

Example: Consider ODE

$$y' = -y \qquad y(0) = 1$$

Exact solution: $y(x) = e^{-x}$

Euler's method:

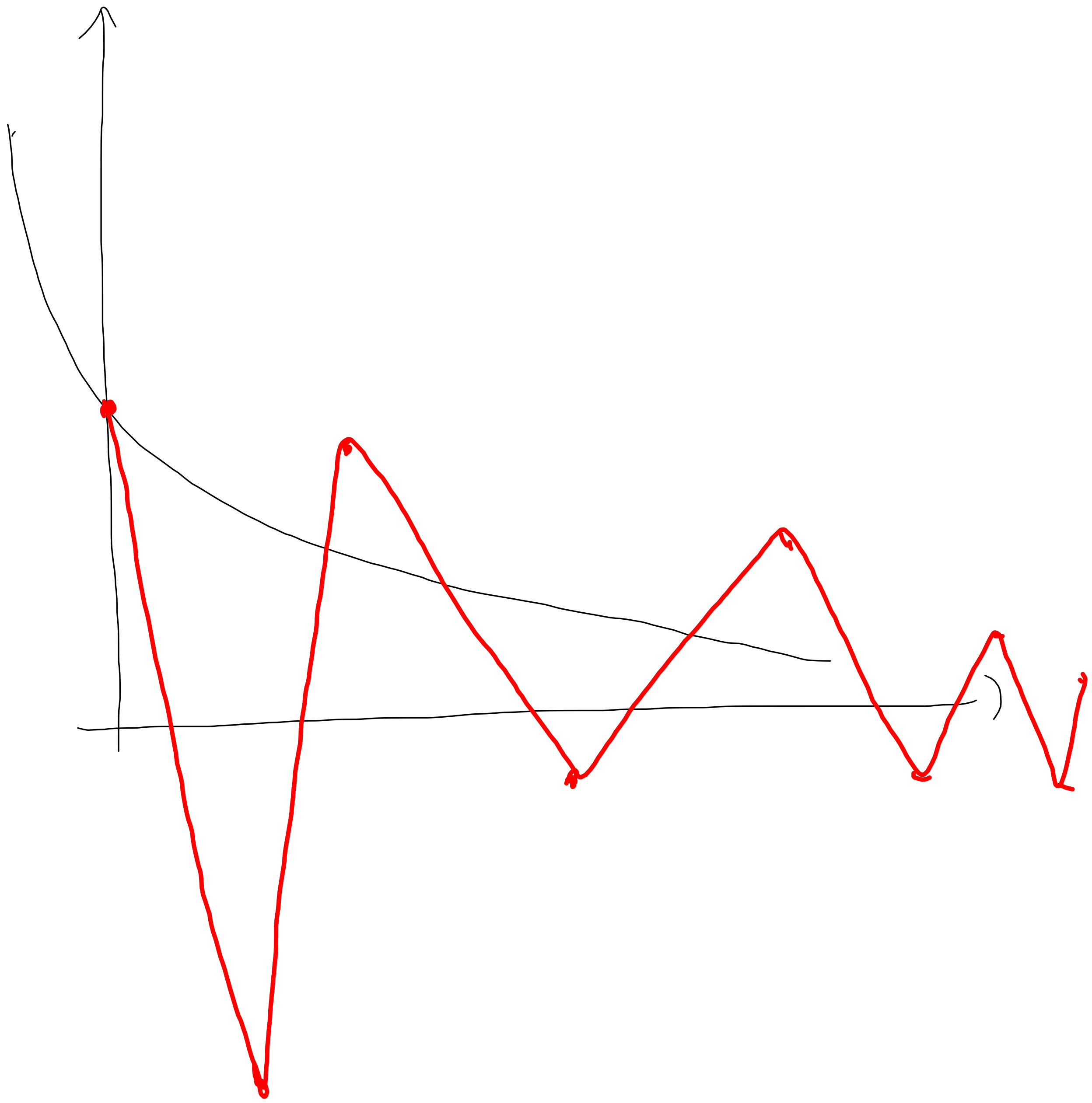
$$y_1 = y(x_0 + h) \simeq y_0 + h\, y'(x_0)$$
$$= y_0 + h(-y_0)$$
$$= 1 - h$$

$$y_2 = y(x_0+2h) = y_1 - h y_1$$
$$= 1-h-h(1-h)$$
$$= (1-h)^2$$

$\vdots$

$$y_n = (1-h)^n$$

If $h > 1$, then the approximation oscillates whereas e^{-x} does not.



Backward Euler's method

$$y_{n+1} = y_n + h(-y_{n+1})$$

$$\Rightarrow \boxed{y_{n+1} = \frac{y_n}{1+h}}$$

$$y_1 = \frac{y_0}{1+h} = \frac{1}{1+h}$$

$$y_2 = \frac{y_1}{1+h} = \frac{1}{(1+h)^2}$$

$$\vdots$$

$$y_n = \frac{1}{(1+h)^n}$$

This does not oscillate for any choice of h.