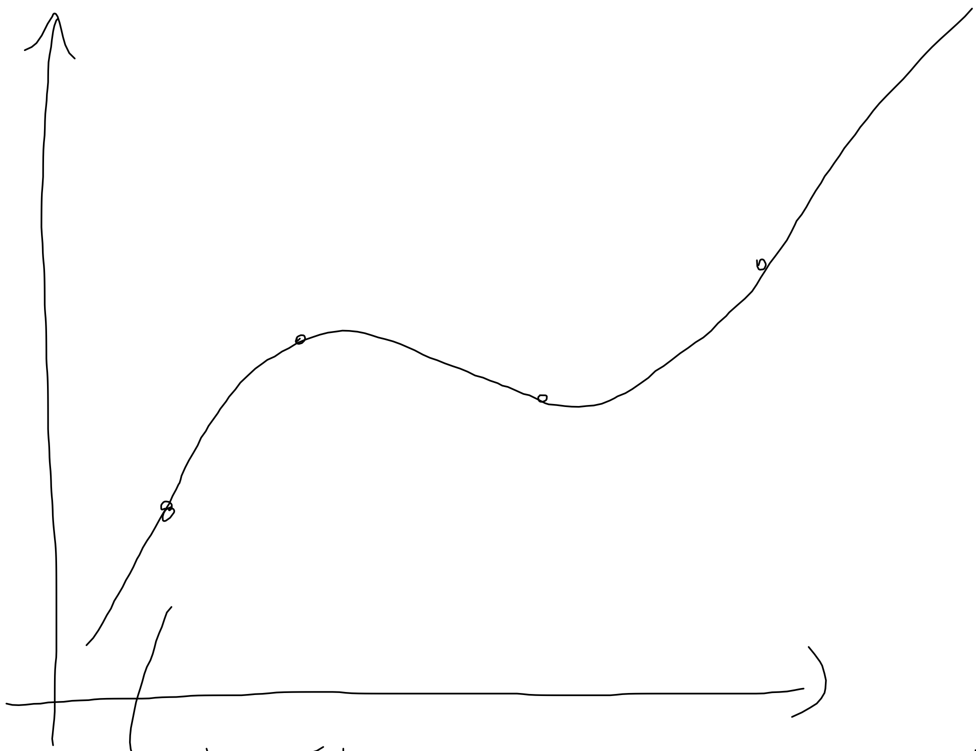
Newton's divided differences

(19.3)



There is a unique polynomial of degree n that goes through n+1 points.

Linear interpolation

(x_0, f_0) & (x_1, f_1)

The line that passes through these points has equation

$$P_1(x) = \frac{f_1 - f_0}{x_1 - x_0} x + \frac{f_0 x_1 - f_1 x_0}{x_1 - x_0}$$

We rewrite:

$$P_1(x) = f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0)$$

$$= f_0 + (x - x_0) f[x_0, x_1]$$

Quadratic interpolation

We add a third point (x_2, f_2). We want

$$P_2(x) = P_1(x) + g(x)$$

degree 2 — degree 2 ?

Have initial conditions:

$$P_2(x_0) = f_0 = P_1(x_0) \Rightarrow g(x_0) = 0$$

$$P_2(x_1) = f_1 = P_1(x_1) \Rightarrow g(x_1) = 0$$

We can factorize

$$g(x) = a_2 (x - x_0)(x - x_1)$$

$$\Rightarrow p_2(x) = p_1(x) + a_2(x - x_0)(x - x_1)$$

We use $p_2(x_2) = f_2$ to find a_2

$$f_2 = p_1(x_2) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$= f_0 + (x_2 - x_0) f[x_0, x_1]$$

$$+ a_2 (x_2 - x_0)(x_2 - x_1)$$

$$\Rightarrow a_2 = \frac{f_2 - f_0 - (x_2 - x_0) f[x_0, x_1]}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$=: f[x_0, x_1, x_2]$$

$$\Rightarrow \boxed{\begin{aligned} P_2(x) = f_0 &+ (x - x_0) f[x_0, x_1] \\ &+ (x - x_0)(x - x_1) f[x_0, x_1, x_2] \end{aligned}}$$

Newton's divided difference:

$$\begin{aligned} P_n(x) = f_0 &+ (x - x_0) f[x_0, x_1] \\ &+ (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2) f[x_0, \dots, x_3] \\ &+ \dots + \end{aligned}$$

$$\cdots + \prod_{k=0}^{n-1}(x-x_k)\, f[x_0, \ldots, x_n]$$

$$\boxed{f[x_0, \ldots, x_k] = \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}}$$

<u>Advantages</u>:

1) Requires less operations
2) It uses previous computations

$$P_n(x) = P_{n-1}(x) + \prod_{k=0}^{n-1}(x-x_k) f[x_0, \dots, x_n]$$

$\uparrow$ interpolation on $x_0, \dots, x_{n-1}$

<u>Example</u>: $f(x) = \sin x$

Find quadratic interpolation using $(1, 0.8415)$, $(2, 0.9093)$, $(3, 0.1411)$

We compute

$$f[x_0, x_1] = \frac{0.9093 - 0.8415}{2-1} = 0.0678$$

$$f[x_1, x_2] = \frac{0.1411 - 0.9093}{3-2}$$

$$= -0.7682$$

	x_k	f_k	$f[x_k, x_{k+1}]$	$f[x_0, x_1, x_2]$
x_0	1	0.8415		
			0.0678	
x_1	2	0.9093		-0.4180
			-0.7682	
x_2	3	0.1411		

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{-0.7682 - 0.0678}{3-1}$$

$$P_2(x) = \underbrace{0.8415 + (x-1)\,0.0678}$$
$$+ (x-1)(x-2) \cdot -0.4180$$

line that goes through $(1, 0.8415)$, $(2, 0.9093)$

Since the polynomial is unique, the interpolation error is exactly the same.

$$\sin 2.5 \approx P_2(2.5) = 0.6297$$

Interpolation at the Chebyshev points

Recall that if $f^{(n+1)}$ exists and is continuous, the error:

$$\varepsilon_n(x) = f(x) - p_n(x)$$

$$= \prod_{k=0}^{n}(x - x_k)\,\frac{f^{(n+1)}(t)}{(n+1)!}$$

for some t in the interval containing x_0, x_n, x.

We want to find $x_0, \ldots, x_n$ that minimizes $\varepsilon_n(x)$ on $[a,b]$. Since t depends on $x_0, \ldots, x_n$, this is hard to do, so we instead minimize

$$\prod_{k=0}^{n} (x - x_k).$$

This is not achieved by taking equidistant points.

We now assume that the points are chosen in $[-1,1]$. We can translate back to points $\tilde{x}_k \in [a,b]$ by change of variables

$$\tilde{x}_k = \frac{(b-a)x_k + (a+b)}{2}.$$

It can be shown that

$$\max_{x\in[-1,1]} \left| \prod_{k=0}^{n} (x-x_k) \right| \geq 2^{-n}.$$

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<u>Chebyshev points</u>

For $x\in[-1,1]$, the n-th <u>Chebyshev polynomial</u>

$T_n(x)$

$$T_n(x) = \cos(n \arccos x)$$

E.g. $T_0(x) = \cos(0) = 1$

$T_1(x) = \cos(\arccos x) = x$

$T_2(x) = \cos(2 \arccos x) = ?$

Trigonometric identity:

$$\cos(n\theta) = 2\cos\theta \cos((n-1)\theta) - \cos((n-2)\theta)$$

$\theta = \arccos x$

$\cos(n \arccos x)$

$= 2\cos(\arccos x)\cos((n-1)\arccos x)$

$- \cos((n-2)\arccos x)$

$$\Rightarrow \boxed{T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x)}$$

$T_2(x) = 2x \cdot x - 1 = 2x^2 - 1$

These are polynomials of degree n s.t.

$$\boxed{\max_{x\in[-1,1]} |T_n(x)| \le 1}$$

since it is a cos

We note that the coefficient in front of x^{n+1} in $T_{n+1}(x)$ is 2^n.
(from recursive formula)

Write

$$T_{n+1}(x) = 2^n (x-\alpha_0)(x-\alpha_2)\cdots(x-\alpha_n)$$

where α_i are the zeros of $T_{n+1}(x)$.

$$\max_{x\in[-1,1]} \left|\prod_{k=0}^{n}(x-\alpha_k)\right|$$

$$= \max_{x\in[-1,1]} \left|2^{-n} T_{n+1}(x)\right| \leq 2^{-n}$$

Since the max of such product can never be less then 2^{-n}

$$\boxed{\Rightarrow \max_{x\in[-1,1]} \left|\prod_{k=0}^{n}(x-\alpha_k)\right| = 2^{-n}}$$

So the product is minimal on $[-1,1]$ choosing $x_k=\alpha_k$, the zeros of $T_{n+1}(x)$.

These are obtained by equating

$$T_{n+1}(x) = \cos\left((n+1)\arccos x\right) = 0$$

$$\Rightarrow \arccos(x_k) = \left(k+\frac{1}{2}\right)\frac{\pi}{n+1}$$

$0 \le k \le n$

$$\Rightarrow \boxed{x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right)}$$

$0 \le k \le n$

These are called
Chebyshev points.

The interpolation error
in $[-1,1]$

zeroes of $T_{n+1}(x)$

$$\varepsilon_n(x) = \prod_{k=0}^{n}(x-x_k)\,\frac{f^{(n+1)}(t)}{(n+1)!}$$

$$\boxed{|\varepsilon_n(x)| \le \frac{2^{-n}}{(n+1)!}\max_{t\in[-1,1]}|f^{(n+1)}(t)| \quad \text{for all } x\in[-1,1]}$$

For the Chebyshev points in $[a,b]$

$$|\varepsilon_n(x)| \leq \frac{2^{-n}}{(n+1)!} \left|\frac{b-a}{2}\right|^{n+1} \max_{t\in[a,b]} |f^{(n+1)}(t)|$$

for all $x \in [a,b]$.

Example: $f(x) = \sin x$.

Find the polynomial of degree 2 which interpolates $f(x)$

at Chebyshev points.
in $[0,4]$.

The roots of $T_3(x)$ are
given by

$$y_k = \cos\left(\frac{2k+1}{6}\pi\right) \quad 0 \leq k \leq 2$$

$\Rightarrow y_0 = -\frac{\sqrt{3}}{2} \quad y_1 = 0, \; y_2 = \frac{\sqrt{3}}{2}$
in $[-1,1]$

We make a change of variables to get the points in [0, 4]

$$x_0 = \frac{1}{2}\left[4\left(-\frac{\sqrt{3}}{2}\right) + 4\right] = -\sqrt{3} + 2$$

$$x_1 = 2 \qquad x_2 = \sqrt{3} + 2$$