

15. 9. 17 TMA 4130

$$u_t = u_{xx}$$

$$u(x, 0) = f(x) = 100 \sin \frac{\pi x}{L}$$

$$u(x, t) \approx \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin \frac{n\pi x}{L}$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx = \begin{cases} 100 & n=1 \\ 0 & n \geq 2 \end{cases}$$

$$\int_0^L \sin \frac{\pi x}{L} \sin \frac{n\pi x}{L} dx =$$

Orthogonalitet av $\left\{ \sin \frac{n\pi x}{L} \right\}_{n \in \mathbb{N}}$

$$\int_0^L \sin \left(\frac{n\pi x}{L} \right) \sin \left(\frac{m\pi x}{L} \right) dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

$$\begin{aligned} u(x, 0) &= B_1 \overset{?}{\sin} \frac{\pi x}{L} + B_2 \overset{?}{\sin} \frac{2\pi x}{L} \\ &+ B_3 \overset{?}{\sin} \frac{3\pi x}{L} + \dots = 100 \\ &\overset{?}{=} 100 \sin \frac{\pi x}{L} \end{aligned}$$

$$\overbrace{\hspace{10em}}^0$$

$$u_t = c^2 u_{xx}$$

$$u(x,0) = f(x) \quad x \in [0, L]$$

$$\underline{u_x(0,t) = u_x(L,t) = 0, t > 0}$$

$$\overbrace{\hspace{10em}}^0 \quad \overbrace{\hspace{10em}}^L$$

$$u(x,t) = ?$$

Separation der Variable:

$$u(x,t) = G(t) F(x)$$

$$\dot{G} F = c^2 G F''$$

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = \text{konst} = -p^2$$

$$\underbrace{\dot{G} = -(cp)^2 G}_{\sim} \quad F'' + p^2 F = 0$$

$$u_x(0,t) = F'(0) G(t) = 0$$

$$u_x(L,t) = F'(L) G(t) = 0$$

$$\Rightarrow F'(0) = F'(L) = 0$$

$$F'' + p^2 F = 0 ; \quad \stackrel{\leftarrow}{F'(0)} = \stackrel{\leftarrow}{F'(L)} = 0$$

$p^2 < 0$: ihren Lösungen

$$p = 0 : \quad F'' = 0 \Rightarrow F = ax + b$$

$$F'(x) = a, \quad F'(0) = F'(L) = 0 \\ \Rightarrow a = 0$$

$P > 0 :$ $F(x) = A \cos Px + B \sin Px$

$$F'(x) = -A P \sin Px + B P \cos Px$$

$$0 = F'(0) = B_p \Rightarrow B = 0$$

$$0 = F'(L) = -A P \sin PL \Rightarrow P = \frac{n\pi}{L} \quad n \in \mathbb{N}$$

Dus: $f_n(x) = \underline{A_n} \cos\left(\frac{n\pi}{L} x\right)$
 $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

Sam fkt:

$$G(t) = G(0) e^{-\lambda_n^2 t}$$

$$\text{des } \lambda_n = c \frac{n\pi}{L} \quad B_n = G(0) A_n$$

Alt da^o:

$$u_n(x, t) = \underline{B_n} e^{-\lambda_n^2 t} \cos \frac{n\pi x}{L}$$

Superposition:

$$u = \sum u_n(x, t) = \sum_{n=0}^{\infty} B_n e^{-\lambda_n^2 t} \cos\left(\frac{n\pi x}{L}\right) \quad t \rightarrow \infty?$$

$$u(x, 0) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right) = f(x) \leftarrow$$

Da ma^o:

1

L

$$\rightarrow B_0 = \frac{1}{L} \int_0^L f(x) dx, B_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

När $t \rightarrow \infty$ vil (intuitivt)

$$u(x,t) \rightarrow B_0 \cos \frac{0\pi x}{L} = B_0 = \text{mådeldverdien av } f.$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

$\xrightarrow[t \rightarrow 0]{}$

Dos $u(x,t) \rightarrow 0$ när $t \rightarrow \infty$

2-dimensionale varmeleddningsligningen (stationære lsn.)

~~$$u_t = c^2 (u_{xx} + u_{yy})$$~~

$u = u(x,y,t)$. Vi skall se på stationäre lsn; dvs $u(x,y,t)$ är uavhängig av t .

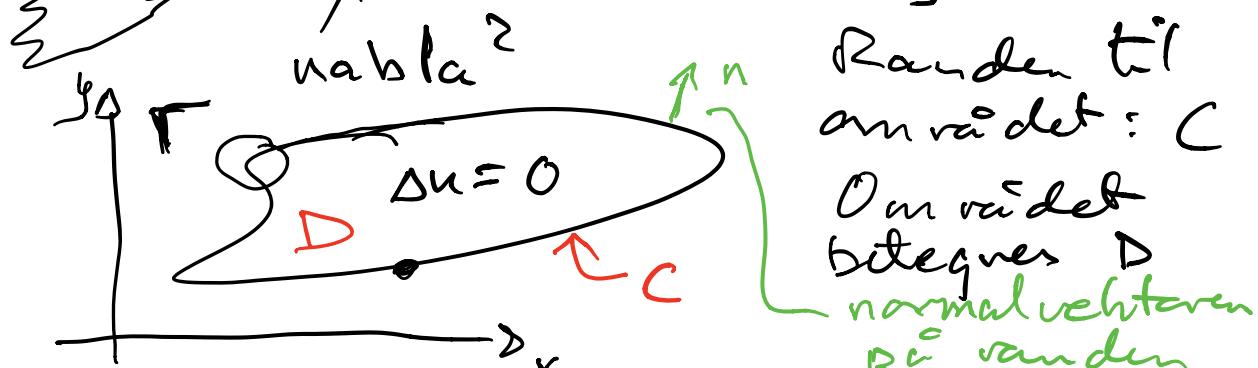
Dvs

$$u_{xx} + u_{yy} = 0$$

Laplaces ligning i 2D.

Laplace- $\rightarrow \Delta u = u_{xx} + u_{yy}$

operatoren $\nabla^2 u = u_{xx} + u_{yy}$



Runden til
område: C
Område
betegnes D
normalvektoren
på vanden

3 mulige rand betingelser: $1 \leq l \leq 1$

① Dirichlet - rand betingelser

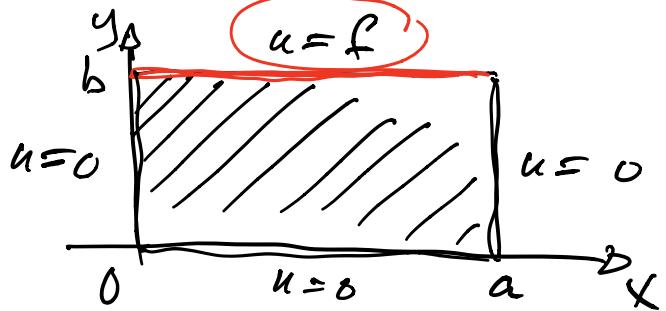
$u = f(x,y)$ for gitt funksjon f
på C

② Neumann - rand betingelser:

$n \cdot \nabla u = \frac{du}{dn} = g(x,y)$ for gitt funksjon g
på C

③ Robin - rand betingelser
(eller mixed - rand betingelser)

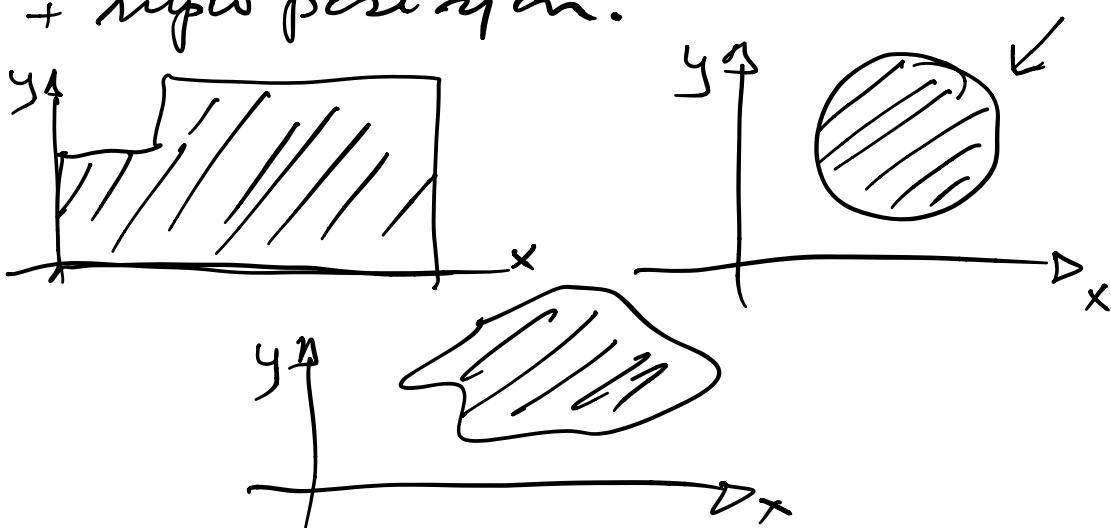
$a u + b \frac{du}{dn} = h(x,y)$ for gitt
 $a = a(x,y)$ funksjon h på C $b = b(x,y)$



$$\Delta u = 0 \quad \text{in } D = [0, a] \times [0, b]$$

Dirichlet randbedingung meint
 $u = 0$ entlang der $y = b$ oder $x \in [0, a]$
 der $u = f$

Separation av variable
 + superposition.



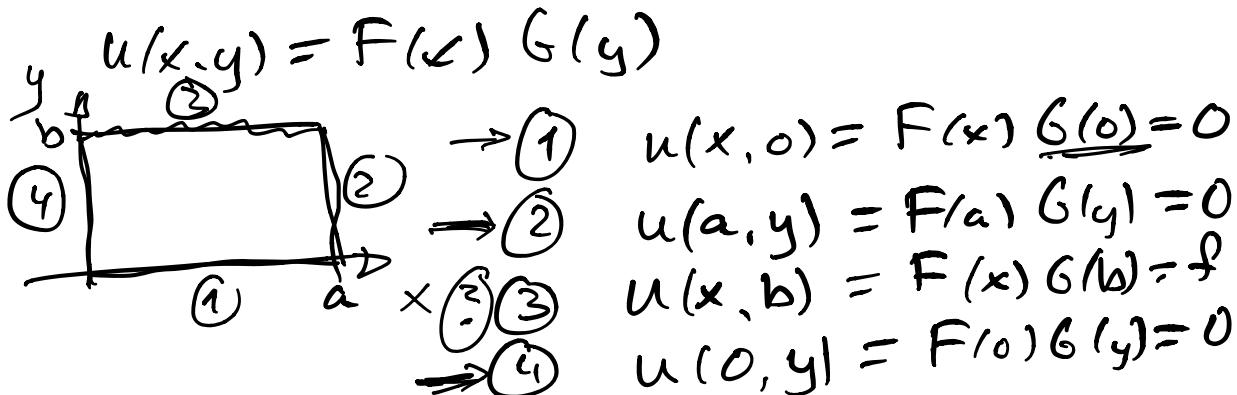
$$u_{xx} + u_{yy} = 0$$

$$u(x, y) = F(x) G(y)$$

$$F_{xx} G + F G_{yy} = 0$$

$$\frac{G_{yy}}{G} = - \frac{F_{xx}}{F} = k$$

$$\rightarrow F_{xx} + kF = 0 , \quad G_{yy} - hG = 0$$



$$\rightarrow F(a) = F(0) = 0$$

Dos: $F_{xx} + kF = 0, \quad F(0) = F(a) = 0$

Da mā $k = \left(\frac{n\pi}{a}\right)^2$ og $\underline{F_n(x) = \tilde{A}_n \sin\left(\frac{n\pi}{a}x\right)}$

$$G_{yy} - hG = 0$$

dos $G_{yy} - \left(\frac{n\pi}{a}\right)^2 G = 0$

Generell løsning: $G(y) = A_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y}$

Randbetingelser krever at $G(0) = 0$

Setter inn:

$$G(0) = A_n + B_n = 0$$

dus

$$G(y) = 2A_n \left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}} \right)$$
$$= A_n \sinh\left(\frac{n\pi y}{a}\right)$$

Dus:

$$u_n(x, y) = F_n(x) G_n(y)$$
$$= A_n^* \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

Superposition:

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y)$$
$$= \sum_{n=1}^{\infty} A_n^* \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

= Fourier-sinusreihen

$$u(x, b) = \sum_{n=1}^{\infty} \overbrace{A_n^* \sinh\left(\frac{n\pi b}{a}\right)}^{\text{Fourier-sinus}} \sin\left(\frac{n\pi x}{a}\right)$$
$$= f(x)$$

dus:

$$\overbrace{A_n^* \sinh\left(\frac{n\pi b}{a}\right)}^a = \text{Fourier-sinus}$$

koeff til f

$$= \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

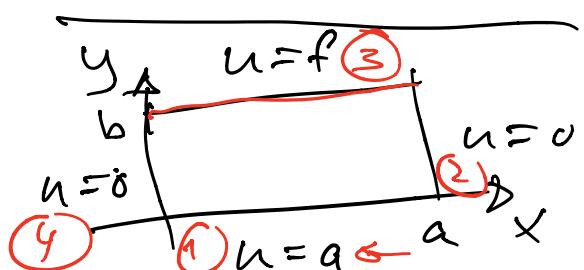
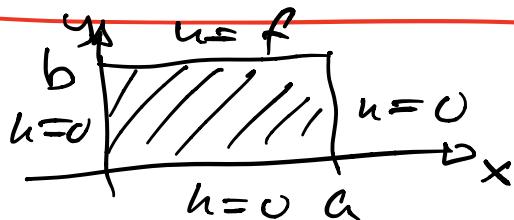
eller

$$\rightarrow A_n^* = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

erg

$$u(x,y) = \sum_{n=1}^{\infty} A_n^* \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

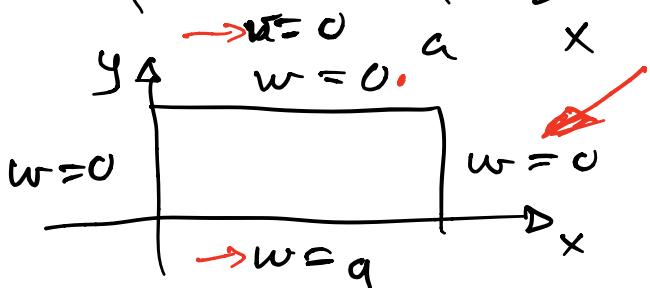
lösbar $\Delta u = 0$



$\Delta u = 0$ ← Randbet



$\Delta v = 0$ ← Randbet



$\Delta w = 0$ ← Randbet

Passtand $u = v + w$

lösbar problem 4

$$\Delta u = \Delta(v + w) = \Delta v + \Delta w = 0$$

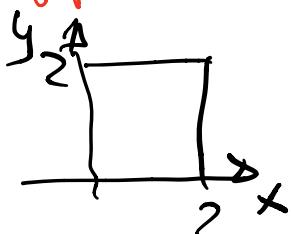
① $u = v + w = 0 + g = g$

② $+ \text{c}_1 u = v + w = 0 + 0 = 0$

③ $u = v + w = f + 0 = f$

thus u lösbar (\mathbb{K})!

Oppg 12.6.19



$$\begin{aligned} \Delta u &= 0 \\ u &= 0 \quad \text{per } \text{vander} \\ \text{untatt } y &= 2 \quad \text{der} \\ u &= 1000 \sin \frac{\pi x}{2} = f(x) \end{aligned}$$

$$u(x,y) = \sum_{n=1}^{\infty} A_n^* \sinh\left(\frac{n\pi y}{2}\right) \sin \frac{n\pi x}{2}$$

$$A_n^* = \frac{1}{2 \sinh(n\pi)} \int_0^2 f(x) \sin \frac{n\pi x}{2} dx$$

$$= \frac{1000}{2 \sinh(n\pi)} \int_0^2 \sin \frac{\pi x}{2} \sin \frac{n\pi x}{2} dx$$

$$\Rightarrow \begin{cases} 0 & n > 1 \\ \frac{1000}{\sinh(\pi)} & n=1 \end{cases}$$

ans.

dss:

$$u(x,y) = 1000 \quad \frac{\sin h(\frac{\pi y}{2})}{\sinh \pi} \sin \frac{\pi x}{2}$$

$$\int_0^2 \sin\left(\frac{\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx = \begin{cases} 1 & m=1 \\ 0 & m \neq 1 \end{cases}$$

speziell Fälle:

$$\int_0^2 \sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{m\pi x}{2}\right) dx = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$$

TMA 4BO 20.10.2012 | m

$$f_1(x_1, \dots, x_n) = 0$$

⋮

$$f_n(x_1, \dots, x_n) = 0$$

$$F = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \quad \underline{F(x) = 0}$$

$\circ(|h|^2)$

Taylor-utvirkning:

$$\rightarrow F(x+h) = F(x) + \underline{\lambda F(x) \cdot h} + \overset{\circ}{\circ}(|h|)$$

der

$$\lambda F = \text{Jacobi-matrisen} = \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{pmatrix}$$

Stor O, liten o notation

$$\underline{f(x)} = \mathcal{O}(g(x)) \text{ når } x \rightarrow a$$

dersom $\left| \frac{f(x)}{g(x)} \right| \leq M \overset{\text{konst.}}{\leftarrow} \text{når } \underline{x \rightarrow a}$

$$\underline{h(x)} = o(\underline{l(x)}) \text{ når } x \rightarrow a$$

dersom

$$\frac{h(x)}{l(x)} \rightarrow 0 \quad \text{når } x \rightarrow a$$

$$f(x) = x^3$$

$$f(x+h) = f(x) + f'(x)h + o(h)$$

$$= \underline{x^3 + 3x^2 h} + o(h)$$

$$f(x+h) = (x+h)^3 = \underline{x^3 + 3x^2 h + 3xh^2 + h^3}$$

$$\text{derned er } o(h) = \underline{3xh^2 + h^3}$$

$$\frac{3xh^2 + h^3}{h} = 3xh + h^2 \xrightarrow[h \rightarrow 0]{} 0$$

$$o(h^2) = 3xh^2 + h^3 ?$$

$$\frac{3xh^2 + h^3}{h^2} \xrightarrow[?]{} 0 = 3x + h$$

Newton's metode :

$$0 = F(x+h) = F(x) + JF(x)h + o(h)$$

$F(x) = 0$ ved iterasjon

Velger $F(x) + JF(x)h = 0$

x_0 . F iinner x_1 ved å løse

$$F(x_0) + JF(x_0)(x_1 - x_0) = 0$$

$$\underline{\underline{JF(x_0)(x_1 - x_0)}} = -F(x_0)$$

$$x_1 - x_0 = -\underline{\underline{JF(x_0)^{-1} F(x_0)}}$$

$$\underline{\underline{x_1 = x_0 - JF(x_0)^{-1} F(x_0)}}$$

Generellt:

$$\underline{\underline{x_{n+1} = x_n - JF(x_n)^{-1} F(x_n)}}$$

Da vil $x_n \rightarrow \bar{x}$ der $F(\bar{x}) = 0$

Velg x_0 ? $JF(x)^{-1}$ dyrt

$$JF(x) \approx 0$$

Kravet er gлатtet av F .

Føllestimat: $\varepsilon_k = x_{n+1} - x_n$

Kan vi se

$$\boxed{\varepsilon_{n+1} = -C \varepsilon_k^2}$$

Eks

$$\begin{aligned} xe^y &= 1 \\ -x^2 + y^2 &= 1 \end{aligned}$$

$$F = \begin{pmatrix} xe^y - 1 \\ -x^2 + y^2 - 1 \end{pmatrix} \quad \leftarrow$$

Skal finne $F(\bar{x}, \bar{y}) = 0$

ved Newtons metode

$$x_{n+1} = x_n - JF(x_n)^{-1} F(x_n)$$

$$\rightarrow \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - JF\left(\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right)^{-1} F\left(\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right)$$

$$JF(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^y & xe^y \\ -2x & 2y \end{pmatrix}$$

$$JF(x, y) = \frac{1}{2ye^y - (-2x)xe^y} \begin{pmatrix} 2y & -xe^y \\ 2x & e^y \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$= \frac{1}{2(y + \cancel{3}x^2)e^y} \begin{pmatrix} 2y & -xe^y \\ 2x & e^y \end{pmatrix}$$

$$= \frac{1}{y + \cancel{3}x^2} \begin{pmatrix} ye^{-y} & -x/2 \\ xe^{-y} & 1/2 \end{pmatrix} \quad \leftarrow$$

Setzt inn:

zweiter

matrix
↓

$$1 v. \dots 1 \quad 1 x_1 \quad , \quad \begin{pmatrix} y_n e^{-y_n} & -x_n/2 \end{pmatrix} \quad \text{Bsp.}$$

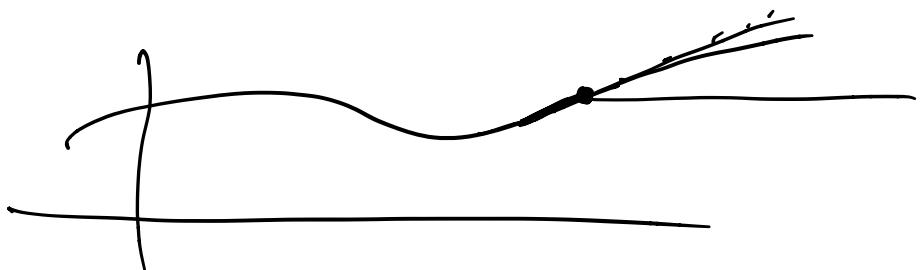
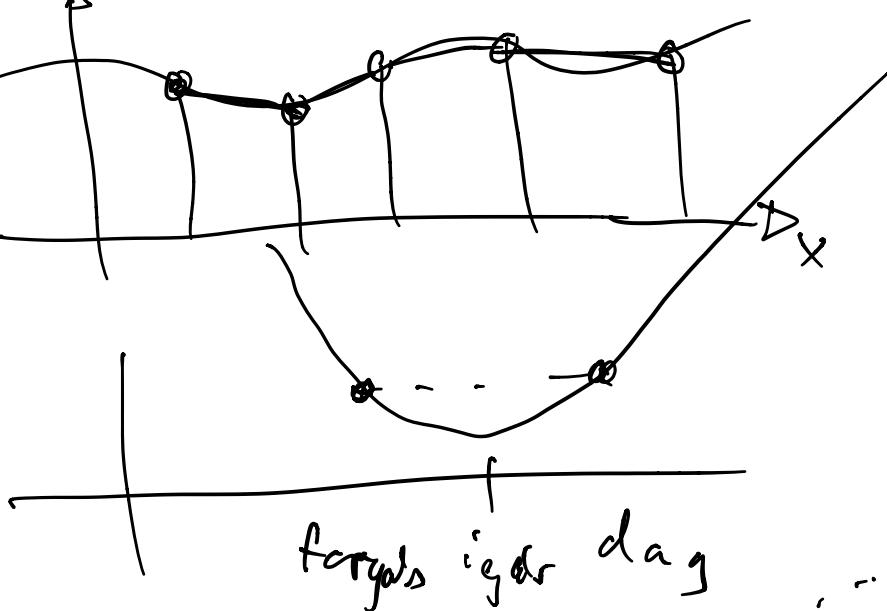
$$\begin{pmatrix} y_{n+1} \\ y_{h+1} \end{pmatrix} = \begin{pmatrix} y_n \\ y_h \end{pmatrix} - \frac{1}{y_n + \cancel{x}_n^2} \begin{pmatrix} \dots & \dots \\ x_n e^{-y_h} & 1/2 \\ \dots & \dots \end{pmatrix} \begin{pmatrix} x_n e^{-y_h} - 1 \\ -x_n^2 + y_n^2 - 1 \end{pmatrix}$$

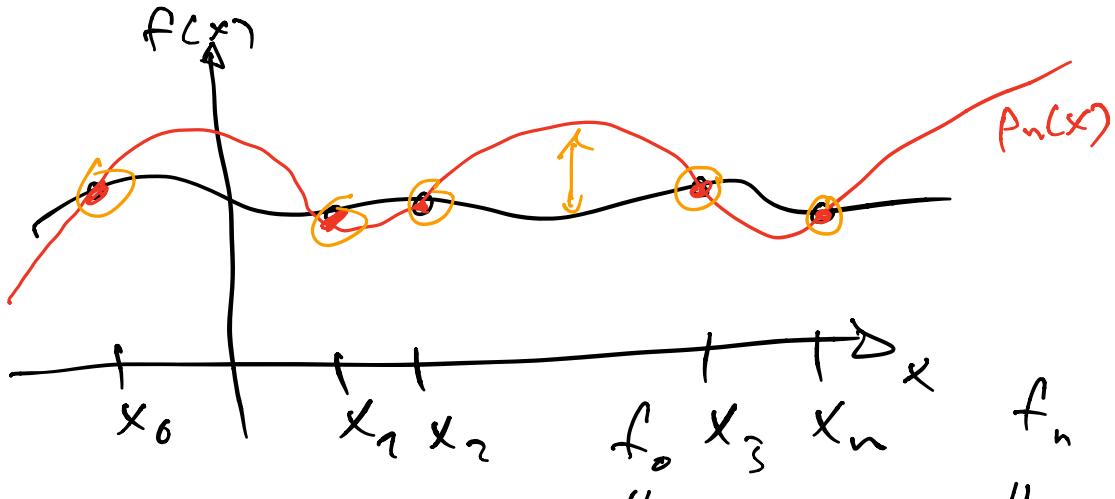
z-vektor \rightarrow

Verg $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Lagrange-Interpolation

$f(x)$





Gitt punkter $(x_0, f(x_0)), \dots, (x_n, f(x_n))$

Finn beste approksimasjon
(interpolasjon) til $y = f(x)$.

Vi skal finne beste polynom
som inter polerer $y = f(x)$.

Weierstrass approksimasyonstetrem.

$f: [a, b] \rightarrow \mathbb{R}$ kontinuerlig.

Gitt $\beta > 0$. Da finns polynomet p_n
av grad n der n avhenger
av β slik at:

$$|f(x) - p_n(x)| < \beta$$

for alle $x \in [a, b]$.

Gitt punktene $(x_0, f_0), \dots, (x_n, f_n)$

Finn polynom p_n av grad n

slik at $\boxed{p_n(x_k) = f_k} \quad k=0, \dots, n$

Dette polynomet er entydig

Anta vi har et annet polynom

q_n slike at $q_n(x_k) = f_k$ for
 $k=0, \dots, n$ grad n .

Definer $r_n = p_n - q_n$ som
er et polynom av grad høyest n .

$$\begin{aligned} \text{slik at } r_n(x_k) &= p_n(x_k) - q_n(x_k) \\ &= f_k - f_k = 0 \end{aligned}$$

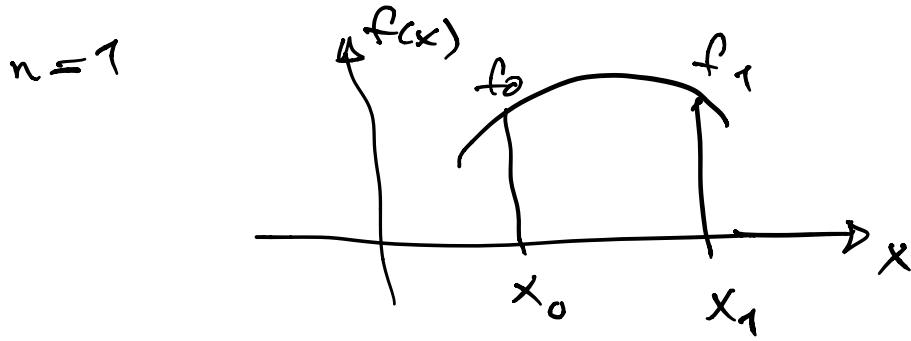
des at $r_n(x_k) = 0$ for $k=0, \dots, n$.

Alt da har r_n $n+1$ nullpunkter,
men har grad høyest n .

Da må $r_n = 0$, des $p_n = q_n$

0

Lagrange-interpolasjon



$$p_1(x) = \underline{Ax + B}$$

$$= L_0 f_0 + L_1 f_1 \leftarrow$$

der L_0, L_1 er polynome 1. grad

1. stückt. at $L_0(x_0) = 1$ erg

$L_0(x_1) = 0$ erg $L_1(x_0) = 0$ erg $L_1(x_1) = 1$

$$p_1(x_0) = f_0$$

$$\rightarrow p_1(x_1) = f_1$$

$$L_0(x) = \underline{\frac{x-x_1}{x_0-x_1}} ; L_1(x) = \underline{\frac{x-x_0}{x_1-x_0}}$$

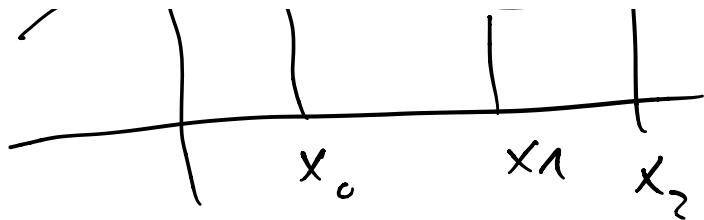
Dan. bilden

$$p_1(x) = L_0(x) f_0 + L_1(x) f_1$$

stückt. at $p_1(x_0) = f_0, p_1(x_1) = f_1$

$n=2$





$$P_2(x) = \underline{L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2}$$

$$L_0(x_0) = 1, \quad L_0(x_1) = L_0(x_2) = 0$$

$$L_1(x_1) = 1, \quad L_1(x_0) = L_1(x_2) = 0$$

$$L_2(x_2) = 1, \quad L_2(x_0) = L_2(x_1) = 0$$

$$\left. \begin{aligned} L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \\ L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \\ L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \end{aligned} \right\} \text{polynome der grad 2}$$

n willkürlich:

V: schreibe:

$$P_n(x) = \sum_{k=0}^n L_k(x) f_k$$

der $L_h(x)$ er polytton av grad
h nikk at

$$L_h(x_k) = 1 \quad \text{og} \quad L_h(x_j) = 0$$

for alle $j \neq k$, des
 $j=0, \dots, h-1, h+1, \dots, n$

Da vil $P_n(x_k) = f_k$ for $k=0, \dots, n$

Definer:

$$l_h(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{h-1})(x-x_{h+1})\dots(x-x_n)}{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{h-1})(x_k-x_{h+1})\dots(x_k-x_n)}$$

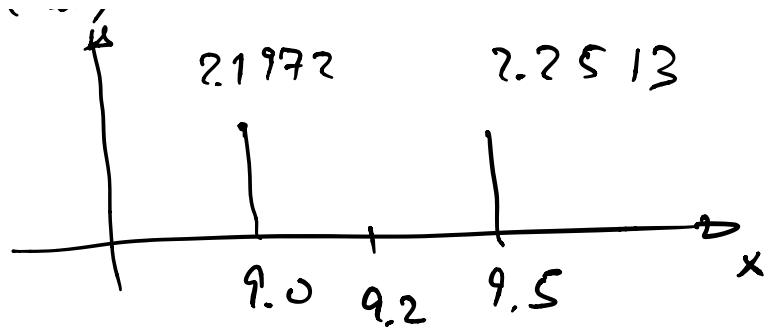
Definer: $L_h(x) = \frac{l_h(x)}{l_h(x_k)}$ ← polytton
av grad n

$$L_h(x_k) = \frac{l_h(x_k)}{l_h(x_k)} = 1$$

$$L_h(x_j) = \frac{l_h(x_j)}{l_h(x_k)} = 0 \quad \text{for } j \neq k$$

Dette er Lagrange interpolasjon

Eks
ved $\ln 9.2$
og $\ln 9.0 = 2.1972$
 Δx $\ln 9.5 = 2.2513$



$$n=1 \quad \left. \begin{aligned} (x_0, f_0) &= (9.0, 2.1972) \\ (x_1, f_1) &= (9.5, 2.2513) \end{aligned} \right\}$$

$$P_1(x) = \frac{L_0(x) f_0 + L_1(x) f_1}{P_1(9.2)} = \dots$$