Convergence order (19.?)

Suppose that you have an iteration method $x_{n+1} = h(x_n)$

- Fixed-point iteration for
  $x = g(x)$
  $\longrightarrow x_{n+1} = g(x_n) =: h(x_n)$
- Newton's method $f(x) = 0$
  $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} =: h(x_n)$$

If $h(x)$ is continuous,
$x_n \to s$, then
s is a fixed-point of h.

$$s = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} h(x_n)$$

continuous $$= h\left(\lim_{n\to\infty} x_n\right) = h(s)$$

The error at each step

$$\varepsilon_n = s - x_n$$

Using Taylor series
around s

$$x_{n+1} = h(x_n) = h(s) + h'(s)(x_n - s) + \frac{1}{2}h''(s)(x_n - s)^2 + \dots$$

Taylor series

$$= h(s) - h'(s)\varepsilon_n + \frac{1}{2}h''(s)\varepsilon_n^2 + \dots$$

(assuming that h is differentiable a number of times)

<u>Definition</u>: The exponent of ε_n in the first non-vanishing term after $h(s)$ is called the <u>Order</u> of the iteration process. It measures the speed of convergence.

Subtract $h(s) = s$ on both sides

$$x_{n+1} - s = -h'(s)\varepsilon_n + \frac{1}{2}h''(s)\varepsilon_n^2 + \dots$$

$$x_{n+1} - s = -\varepsilon_{n+1}$$

a) If $h'(s) \neq 0$ (order 1)

$$\varepsilon_{n+1} \approx h'(s)\varepsilon_n = C\varepsilon_n$$

(Since the other terms are very small compared to the first term)

b) If $h'(s)=0$, but $h''(s)\neq 0$ (order 2)

$$-\varepsilon_{n+1} \approx \frac{1}{2}h''(s)\,\varepsilon_n^2 = C\,\varepsilon_n^2$$

This means that if $\varepsilon_n = 10^{-k}$, then $\varepsilon_{n+1} = C\,10^{-2k}$

so the number of significant digits is doubling.

Convergence for Newton's method

$$f(x) = 0$$

$$h(x) = x - \frac{f(x)}{f'(x)}$$

$$h'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2}$$

$$= \frac{f(x)f''(x)}{f'(x)^2}$$

$$h'(s) = \frac{\underbrace{f(s)}_{=0} f''(s)}{f'(s)^2} = 0$$

$$h''(s) = \frac{f''(s)}{f'(s)} \neq 0 \quad \text{(in general)}$$

$\Longrightarrow$ Convergence of order 2.

(However, have slow convergence if $f'(s) \approx 0$)

- Need f to be 3 times differentiable.

$$\varepsilon_{n+1} = \underbrace{-\frac{1}{2}\frac{f''(s)}{f'(s)}}_{C} \varepsilon_n^2$$

Example: Give a prior estimate (only computing 1st iteration) for the number of steps required to find a solution to $\sin x - e^{-x} = 0$ with 5 decimals accuracy.

Last time $x_0 = 1$

$x_1 = 0.4785$

WANT error $< 5 \cdot 10^{-6}$

$$f'(x) = \cos x + e^{-x}$$

$$f''(x) = -\sin x - e^{-x}$$

$$\Rightarrow C = -\frac{1}{2}\frac{f''(s)}{f'(s)} \approx -\frac{1}{2}\frac{f''(x_1)}{f'(x_1)}$$

$$= 0.4510$$

Hence;

$$|\varepsilon_{n+1}| \approx 0.4510\,|\varepsilon_n|^2$$

$$\approx 0.4510 \cdot \left(0.4510 \cdot |\varepsilon_{n-1}|^2\right)^2$$

$$= 0.4510^3 \, |\varepsilon_{n-1}|^4$$

$$\approx 0.4510^3 \left(0.4510 \cdot |\varepsilon_{n-2}|^2 \right)^4$$

$$= 0.4510^7 \, |\varepsilon_{n-2}|^8$$

$$\vdots$$

$$\approx 0.4510^M \, |\varepsilon_0|^{M+1}$$

where $M = 1 + 2 + 2^2 + \ldots + 2^n$

$$= 2^{n+1} - 1$$

We now give an estimate on $\varepsilon_0 = s - x_0 = s - 1$

$$\varepsilon_1 - \varepsilon_0 = (\varepsilon_1 - s) - (\varepsilon_0 - s)$$
$$= -x_1 + x_0$$
$$= 0.5215$$

$$0.4510\,\varepsilon_0^2 \approx \varepsilon_1 = \varepsilon_0 + 0.5215$$

$$\Rightarrow -0.4510\,\varepsilon_0^2 + \varepsilon_0 + 0.5215 \approx 0$$

$$\Rightarrow \varepsilon_0 \approx -0.4358$$

$\cancel{\varepsilon_0 \approx 2.6}$ (since we know $s - 1 < 1$)

$$\Rightarrow |\varepsilon_{n+1}| = 0.4510^{M}\; 0.4358^{M+1} \underset{\text{WANT}}{<} 5\cdot 10^{-6}$$

Solving the inequality

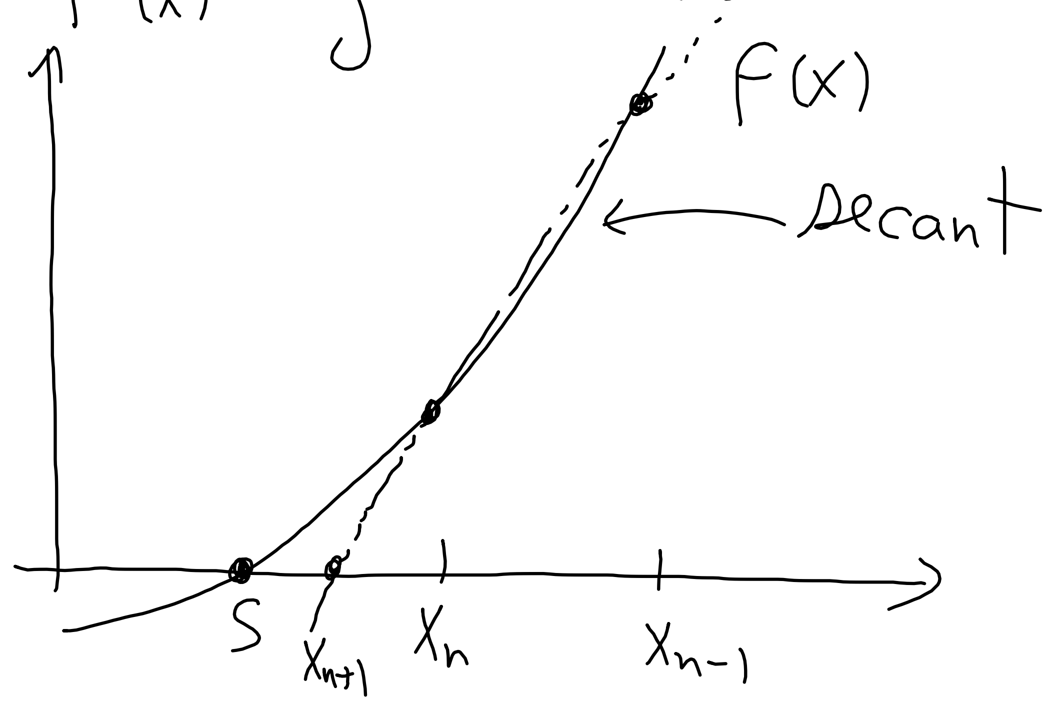
$$\underset{\underset{2^{n+1}-1}{\shortparallel}}{M} > \frac{\ln(5\cdot 10^{-6}/0.4358)}{\ln(0.4510\cdot 0.4358)} = 6.9$$

$\Rightarrow n \geq 2.$

$\Rightarrow$ We estimate that x_3 is accurate to 5 decimals.

Secant method for solving $f(x) = 0$

If the derivative is very small, can replace $f'(x)$ by a secant.



Start with x_0 & x_1

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

zero of the secant.

Order of convergence

$$\varepsilon_{n+1} \approx C \cdot \varepsilon_n^{1.62}$$

Be careful!

$$\xcancel{x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}}$$

You risk losing significant digits.

Newton's method in higher dimension

(Notes online)

The goal is to solve a system of equations of the form

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ f_2(x_1, \dots, x_n) = 0 \\ \vdots \\ f_n(x_1, \dots, x_n) = 0 \end{cases}$$

$$f_j : \mathbb{R}^n \longrightarrow \mathbb{R}$$

We write $\vec{x} = (x_1, \dots, x_n)$

The equation becomes

$$F(\vec{x}) := \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{pmatrix} = \vec{0}$$

$F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

The system is not necessarily linear, so the idea is to approximate it by a linear system and solve it.

Def: The Jacobian of F at a point $\vec{x} \in \mathbb{R}^n$ is defined as

$$JF(\vec{x}) = \begin{pmatrix} \partial_1 f_1(\vec{x}) & \partial_2 f_1(\vec{x}) & \cdots & \partial_n f_1(\vec{x}) \\ \partial_1 f_2(\vec{x}) & \partial_2 f_2(\vec{x}) & \cdots & \partial_n f_2(\vec{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 f_n(\vec{x}) & \partial_2 f_n(\vec{x}) & \cdots & \partial_n f_n(\vec{x}) \end{pmatrix}$$

This gives a <u>linear approximation</u> of F.
(Taylor series)

If $F: \mathbb{R}^n \to \mathbb{R}^n$ is twice continuously differentiable

$$(1) \quad F(\vec{x}) \approx F(\vec{x}_0) + JF(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

for every $\vec{x}, \vec{x}_0 \in \mathbb{R}^n$.

"notion of derivatives"

Assume that $\vec{s}$ is a solution to $F(\vec{x}) = \vec{0}$.

$$(1) \quad \underset{\vec{0}}{\underset{\parallel}{F(\vec{s})}} \approx F(\vec{x}_0) + JF(\vec{x}_0)(\vec{s} - \vec{x}_0)$$

$$\Rightarrow \vec{s} \approx \vec{x}_0 - JF^{-1}(\vec{x}_0) F(\vec{x}_0)$$

Newton's Method:

$$\boxed{\vec{X}_{n+1} = \vec{X}_n - JF^{-1}(\vec{x}_n)F(\vec{x}_n)}$$

Need JF to be invertible ($\det \neq 0$)

Problems arise when JF is very closed to be non-invertible around $\vec{s}$. ($\det JF(\vec{s}) \approx 0$)

Also computing JF^{-1} Can be hard.

One could also compute the solution to the equation

$$JF(\vec{x}_n)\vec{h} = -F(\vec{x}_n)$$

and write $\vec{x}_{n+1} = \vec{x}_n + \vec{h}$

$\rightarrow$ coming from Taylor's series.

(This is a system of linear equations).