Convolution $*$

$$(f * g)(t) = \int_0^t f(\alpha)\, g(t-\alpha)\, d\alpha$$

$$\mathcal{L}(f * g) = \mathcal{L}(f)\,\mathcal{L}(g)$$

Example: $y'' + 3y' + 2y = r$

$$r(t) = \begin{cases} 1 & \text{if } 1 < t < 2 \\ 0 & \text{else} \end{cases}$$

$y(0) = 0 \quad y'(0) = 0$

$$\mathcal{L}(y) = Q(s)\,\mathcal{L}(r)$$

where $Q = \dfrac{1}{s^2+3s+2}$

$$\Rightarrow y = q(t) * r(t)$$

where $q(t) = \mathcal{L}^{-1}(Q)$

$$= e^{-t} - e^{-2t}$$

$$y(t) = \int_0^t q(t-\alpha)\, r(\alpha)\, d\alpha$$

If $t < 1$,

$$y(t) = \int_0^t q(t-\alpha)\,r(\alpha)\,d\alpha = 0$$

since r is 0 for $t < 1$.

If $1 < t < 2$,

$$y(t) = \int_1^t q(t-\alpha)\cdot 1\,d\alpha$$

$$= \int_1^t \left[e^{-(t-\alpha)} - e^{-2(t-\alpha)}\right] d\alpha$$

$$= \frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}$$

If $t > 2$

$$y(t) = \int_1^2 g(t-\alpha) \cdot 1 \, d\alpha$$

$$= \int_1^2 \left[e^{-(t-\alpha)} - e^{-2(t-\alpha)} \right] d\alpha$$

$$= e^{-(t-2)} - \frac{1}{2} e^{-2(t-2)} - e^{-(t-1)}$$

$$+ \frac{1}{2} e^{-2(t-1)}$$

Integral equations

Example: Solve

$$y(t) - \int_0^t y(\alpha)\sin 2(t-\alpha)\,d\alpha = \sin 2t$$

This is a convolution:

$$y(t) - y(t) * \sin 2t = \sin 2t$$

Applying LT:

$$\mathcal{L}(y) - \mathcal{L}(y)\,\mathcal{L}(\sin 2t) = \mathcal{L}(\sin 2t)$$

Thus: $\mathcal{L}(y) = \dfrac{\mathcal{L}(\sin 2t)}{1-\mathcal{L}(\sin 2t)}$

$$= \frac{\frac{2}{s^2+4}}{1-\frac{2}{s^2+4}} = \frac{2}{s^2+2} = \sqrt{2}\cdot\frac{\sqrt{2}}{s^2+2}$$

Applying inverse LT gives

$$\boxed{y(t) = \sqrt{2}\,\sin\sqrt{2}t}$$

Differentiation & integration of LT (6.6)

$$\mathcal{L}(t\,f(t)) = -\frac{d}{ds}\left(\mathcal{L}(f)(s)\right)$$

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty \mathcal{L}(f)(\sigma)\,d\sigma$$

Example: Find LT of $t^2 \sin 3t$.

$$\mathcal{L}(t^2 \sin 3t) = -\frac{d}{ds}\mathcal{L}(t \sin 3t)$$

$$= \frac{d^2}{ds^2}\mathcal{L}(\sin 3t)$$

$$= \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right) = \frac{18(s^2-3)}{(s^2+9)^3}$$

Example: Find inverse LT of $\ln \frac{s}{s-1}$

$$\mathcal{L}^{-1}\left(\ln \frac{s}{s-1}\right) = \mathcal{L}^{-1}\left(-\int_s^\infty \frac{1}{\sigma(1-\sigma)}\,d\sigma\right)$$

$$= -\frac{\mathcal{L}^{-1}\left(\frac{1}{s(1-s)}\right)}{t}$$

Integration of LT

$$= -\frac{\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s-1}\right)}{t} = \frac{e^{t} - 1}{t}.$$

Example: Laguerre's ODE

$$t y'' + (1-t) y' + n y = 0$$

n integer ≥ 0

Applying LT

$$\mathcal{L}(ty'') + \mathcal{L}(y') - \mathcal{L}(ty') + n\mathcal{L}(y) = 0$$

$$\Rightarrow -\frac{d}{ds}\mathcal{L}(y'') + [s\mathcal{L}(y) - y(0)] - \left(-\frac{d}{ds}\mathcal{L}(y')\right) + n\mathcal{L}(y) = 0$$

$$\Rightarrow -\frac{d}{ds}[s^2\mathcal{L}(y) - sy(0) - y'(0)] + [s\mathcal{L}(y) - y(0)] + \frac{d}{ds}[s\mathcal{L}(y) - y(0)] + n\mathcal{L}(y) = 0$$

$$\Rightarrow -2s\mathcal{L}(y) - s^2\frac{d}{ds}\mathcal{L}(y) + \cancel{y(0)}$$

$$+ s\mathcal{L}(y) - \cancel{y(0)} + \mathcal{L}(y) + s\frac{d}{ds}\mathcal{L}(y)$$

$$+ n\mathcal{L}(y) = 0$$

$$\Rightarrow (s-s^2)\frac{d}{ds}\mathcal{L}(y) + (n+1-s)\mathcal{L}(y) = 0$$

This is an ODE of $\mathcal{L}(y)$ as a function of s.

$Y = \mathcal{L}(y)$ Using separation of variables

$$\int \frac{dY}{Y} = \int -\frac{n+1-s}{s-s^2}\, ds$$

$$= \int \left(\frac{n}{s-1} - \frac{n+1}{s} \right) ds$$

$$\Rightarrow \ln|Y| = \ln|s-1|^n - \ln|s|^{n+1} + C$$

$$\Rightarrow Y = \boxed{K \frac{(s-1)^n}{s^{n+1}} = \mathcal{L}(y)}$$

We take $K = 1$.

In order to find $\mathcal{L}^{-1}$,
we note that

$$\mathcal{L}\left(t^n e^{-t}\right) = -\frac{d^n}{ds^n}\,\frac{1}{s+1}$$

$$= \frac{n!}{(s+1)^{n+1}}$$

and

$$\mathcal{L}\left(\frac{d^n}{dt^n}\left(t^n e^{-t}\right)\right) = \frac{n!\,s^n}{(s+1)^{n+1}}$$

$$\Rightarrow \mathcal{L}\left(\frac{e^t}{n!}\,\frac{d^n}{dt^n}\left(t^n e^{-t}\right)\right) = \frac{(s-1)^n}{(s)^{n+1}}$$

$$\Rightarrow \boxed{y(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\left(t^n e^{-t}\right)}$$

↳ Laguerre polynomials

$n=0$: $y(t) = 1$

$n=1$: $y(t) = -t+1$

$n=2$: $y(t) = \frac{1}{2}\left(t^2 - 4t + 2\right)$

Numerical Analysis (19)

Error of numeric results

An approximation is only useful is we understand/control the error that we make.

Error $= \varepsilon$

$$\boxed{\varepsilon = a - \tilde{a}}$$

$\tilde{a}$ → approximation

a → exact value

$$a = \tilde{a} + \varepsilon$$

true value = approx. + error

Definition: The relative error is defined

$$\varepsilon_r = \frac{\varepsilon}{a}$$

We never know a, and so we can estimate it by $\varepsilon_r \approx \frac{\varepsilon}{\tilde{a}}$

(Good if $|\varepsilon|$ is much less than $|\tilde{a}|$)

But we don't know ε either, so we instead look at bounds.

Definition: The error bound is a number β s.t.

$$|\varepsilon| \leq \beta \Rightarrow |a - \tilde{a}| \leq \beta$$

The relative error bound

$$|\varepsilon_r| \leq \beta_r.$$

Every time we make an approximation, we make a mistake. We want to know how a mistake propagates.

Theorem: 1. In addition or substraction if
β_1 is a bound for ε_1
β_2 " " " ε_2

Then $\beta_1+\beta_2$ is a bound for the error on $\tilde{a}_1+\tilde{a}_2$
$\tilde{a}_1-\tilde{a}_2$.

Proof: ε error on a_1+a_2

$$|\varepsilon| = |(a_1+a_2)-(\tilde{a}_1+\tilde{a}_2)|$$
$$= |(a_1-\tilde{a}_1)+(a_2-\tilde{a}_2)|$$

$$\leq |(a_1 - \tilde{a}_1)| + |a_2 - \tilde{a}_2|$$

$$\leq \beta_1 + \beta_2$$

2. In multiplication or division, the <u>relative error</u> on $\tilde{a}_1 \tilde{a}_2$ or $\tilde{a}_1 / \tilde{a}_2$ is given by the <u>sum</u> $\beta_{1r} + \beta_{2r}$.

Loss of significant digits

It means that the approximation has loss significant digits than the numbers it was obtained from.

It happens when there is a cancellation of two similar numbers.

<u>Example</u>: Find the roots of $x^2+40x+2=0$ using 4 significant digits.

$$x_1 = \frac{1}{2a}\left(-b+\sqrt{b^2-4ac}\right)$$

$$x_2 = \frac{1}{2a}\left(-b-\sqrt{b^2-4ac}\right)$$

We get

$$x_1 = -20.00 + 19.95$$
$$= -0.05$$

This is <u>bad</u> because you started with 4 sign. dig. and ended it up with 1.

$$x_2 = -20.00 - 19.95 = -39.95 \quad \checkmark$$

Instead,

$$x_1 = \frac{c}{a x_2} \qquad x_2 = \text{same}$$

Then $x_1 = 2.000/(-39.95)$

$= -0.05006.$

Sometimes, there is no good formula for the bound β of an error.

The strategy is then to perform the algorithm with N_1 and N_2 iterations and estimate the error in the first approximation

$$\boxed{\varepsilon_{N_1} \approx \tilde{a}_{N_2} - \tilde{a}_{N_1}}$$

In fact: real value

$$\tilde{a}_{N_1} + \varepsilon_{N_1} = \tilde{a}_{N_2} + \varepsilon_{N_2}$$

$$\Rightarrow \tilde{a}_{N_1} - \tilde{a}_{N_2} = \varepsilon_{N_1} - \varepsilon_{N_2} \approx \varepsilon_{N_1}$$

$\Rightarrow$ Since the algorithm converges, the error is getting smaller, and so ε_{N_2} is small compared to ε_{N_1}.