Laplace Transform for ODEs

output input

IVP $y'' + ay' + by = r(t)$

$y(0) = K_0 \quad y'(0) = K_1$

$$\mathcal{L}(y) = \frac{(s+a)y(0) + y'(0)}{s^2 + as + b} + \frac{\mathcal{L}(r)}{s^2 + as + b}$$

We call $\dfrac{1}{s^2+as+b}$ "$Q(s)$" the

transfer function.

If $y(0) = y'(0) = 0$, then

$$\frac{1}{Q(s)} = \frac{\mathcal{L}(r)}{\mathcal{L}(y)} = \frac{\mathcal{L}(\text{input})}{\mathcal{L}(\text{output})}$$

It only depends on a and b. (not r)

<u>Example</u>: Solve the ODE

$$y'' + 9y = e^{-t} \qquad y(0) = 0, \quad y'(0) = 0$$

Apply LT:

$$s^2\mathcal{L}(y) + 9\mathcal{L}(y) = \mathcal{L}(e^{-t})$$

$$\Rightarrow \mathcal{L}(y) = \frac{\mathcal{L}(e^{-t})}{s^2+9} \quad \text{☆}$$

$$= \frac{\frac{1}{s+1}}{s^2+9} = \frac{1}{(s+1)(s^2+9)}$$

We just need to apply the inverse LT, but in order to use the tables (p.249-251 or website) we need to transform ☆ in a form that we know.

Partial fractions

$$\frac{1}{(s+1)(s^2+9)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+9}$$

$$= \frac{A(s^2+9) + (Bs+C)(s+1)}{(s+1)(s^2+9)}$$

$$\Rightarrow \quad 1 = As^2 + 9A + Bs^2 + Bs + Cs + C$$

$$\Rightarrow \quad 1 = (A+B)s^2 + (B+C)s + (9A+C)$$

$$\begin{cases} (A+B) = 0 \\ B+C = 0 \\ 9A+C = 1 \end{cases} \Rightarrow \quad \begin{aligned} A &= 1/10 \\ B &= -1/10 \\ C &= 1/10 \end{aligned}$$

$$\frac{1}{(s+1)(s^2+9)} = \frac{1/10}{s+1} + \frac{-1/10\, s + 1/10}{s^2+9}$$

$$= \frac{1}{10}\cdot\frac{1}{s+1} - \frac{1}{10}\,\frac{s}{s^2+3^2}$$

$$+ \frac{1}{10}\cdot\frac{1}{s^2+3^2}$$

Applying the inverse LT

$$y = \frac{1}{10}e^{-t} - \frac{1}{10}\cos 3t + \frac{1}{30}\sin 3t$$

TABLES p. 249   7   14   13

Example $y'' - y = 50t - 100$
$= 50(t-2)$

$y(2) = 1 \qquad y'(2) = 1$

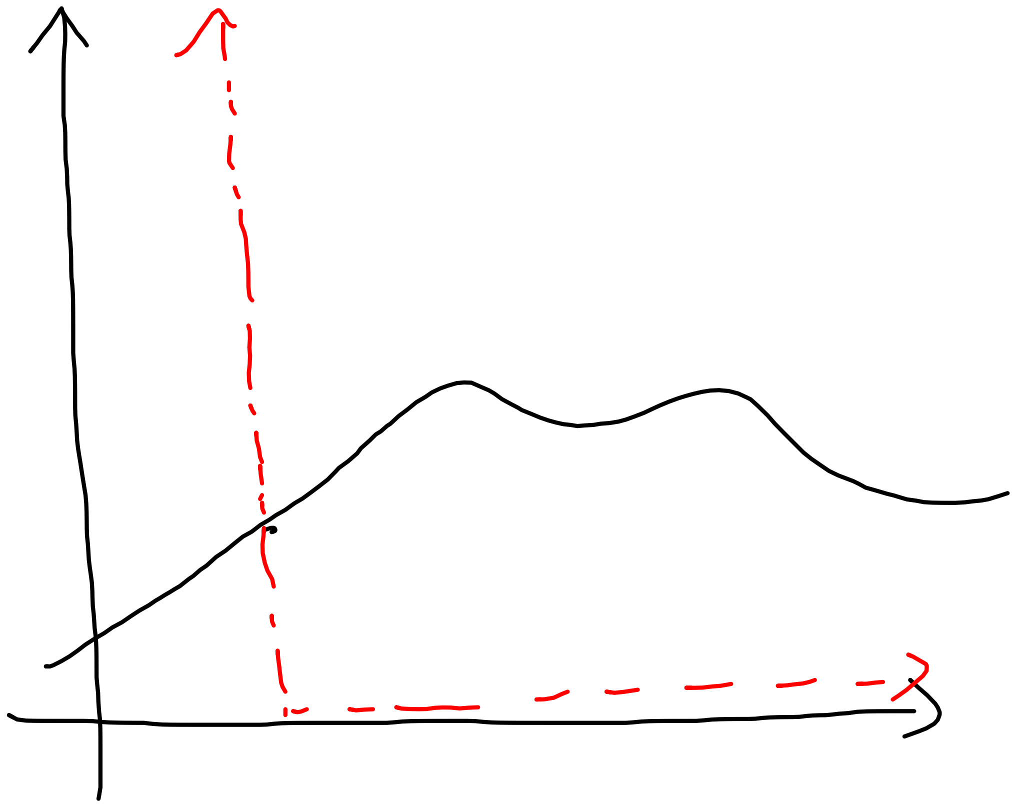
Cannot apply LT directly because the formula for derivatives require $y(0)$ & $y'(0)$.

Strategy: $\tilde{t} = t - 2 \quad \tilde{y} = y(\tilde{t})$

Then the ODE becomes

$$\tilde{y}'' - \tilde{y} = 50(t-2) = 50\tilde{t}$$

$$\tilde{y}(0) = 1 \qquad \tilde{y}'(0) = 1$$



Apply the LT

$$\mathcal{L}(\tilde{y}) = \frac{s+1}{s^2-1} + \frac{50}{s^2(s^2-1)}$$

$$\left(\mathcal{L}(50\tilde{t}) = \frac{50}{s^2} \right)$$

$$= \frac{\cancel{s+1}}{\cancel{(s+1)}(s-1)} + \frac{50}{s^2(s^2-1)}$$

$$= \frac{1}{s-1} + \frac{50}{s^2-1} - \frac{50}{s^2}$$

Apply inverse LT

$$= e^{\tilde{t}} + 50\sinh\tilde{t} - 50\tilde{t}$$

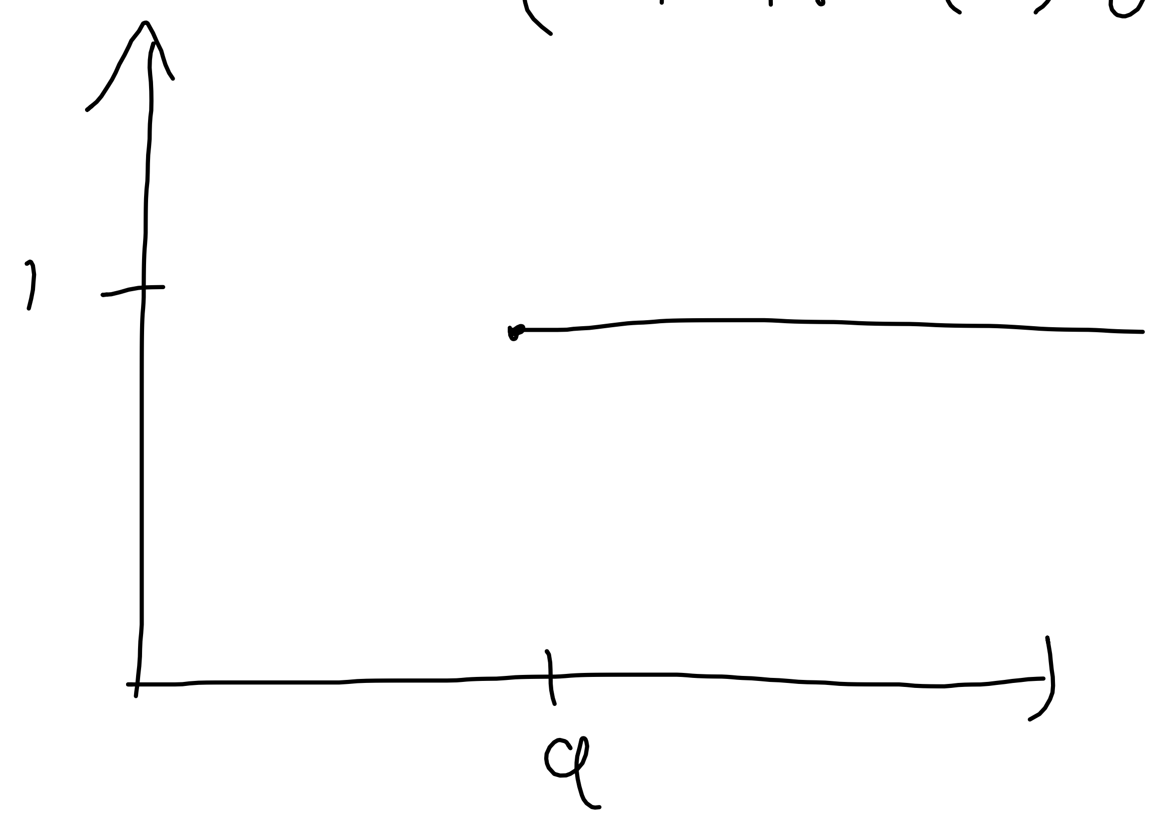
$$(7 \qquad 15 \qquad 2)$$

$$= \boxed{e^{t-2} + 50\sinh(t-2) - 50(t-2)}$$

Unit step Function (6.3)

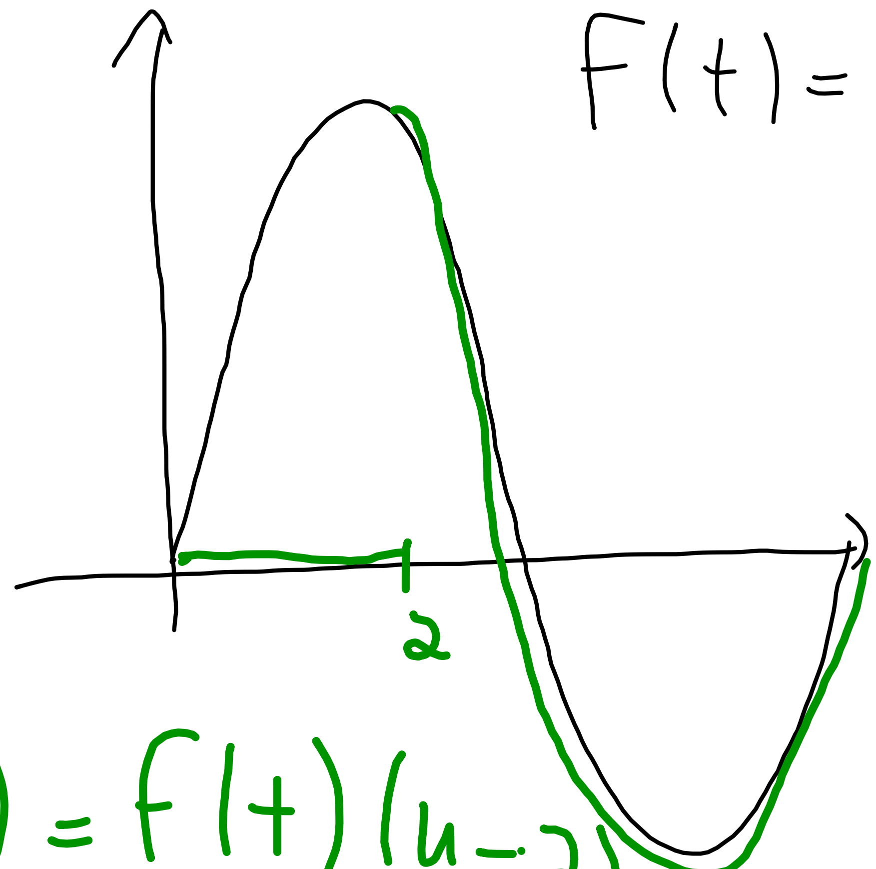
Definition: The unit step function or Heaviside function is

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$



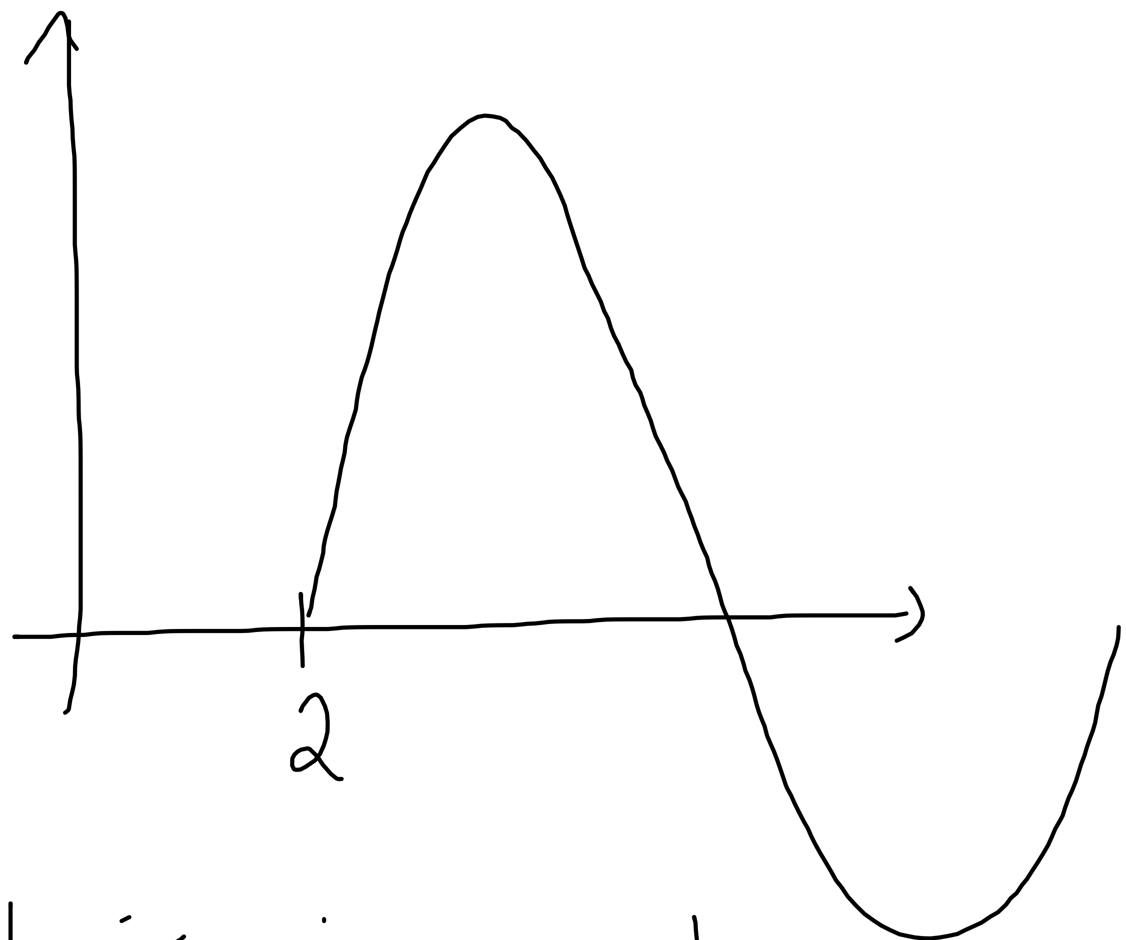
Intuitively, it represents situations in which the involved functions can be switched "on" or "off".

Ex.: $f(t) = \sin t$



$g(t) = f(t)(u-2)$

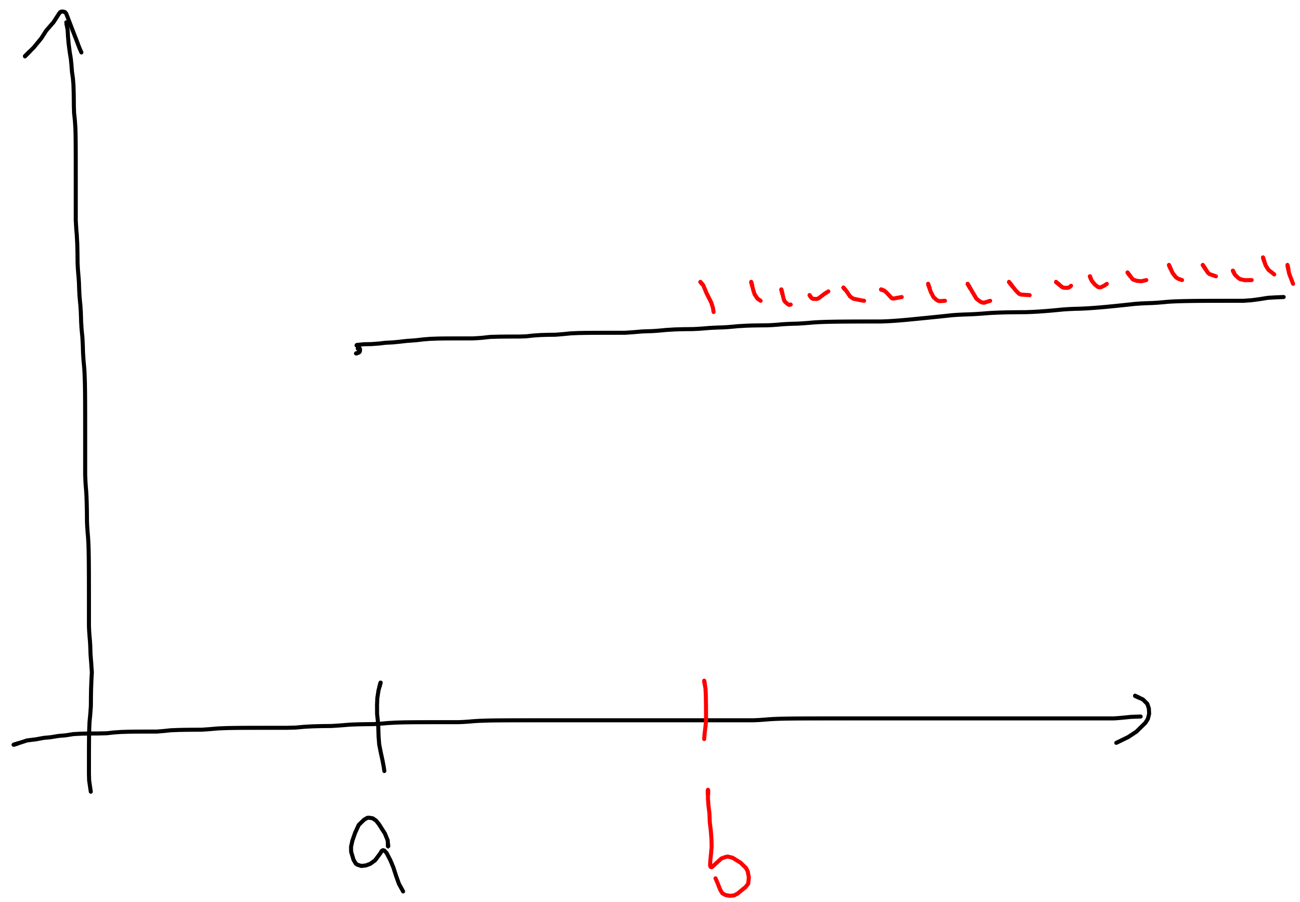
$$h(t) = f(t-2)(u-2)$$



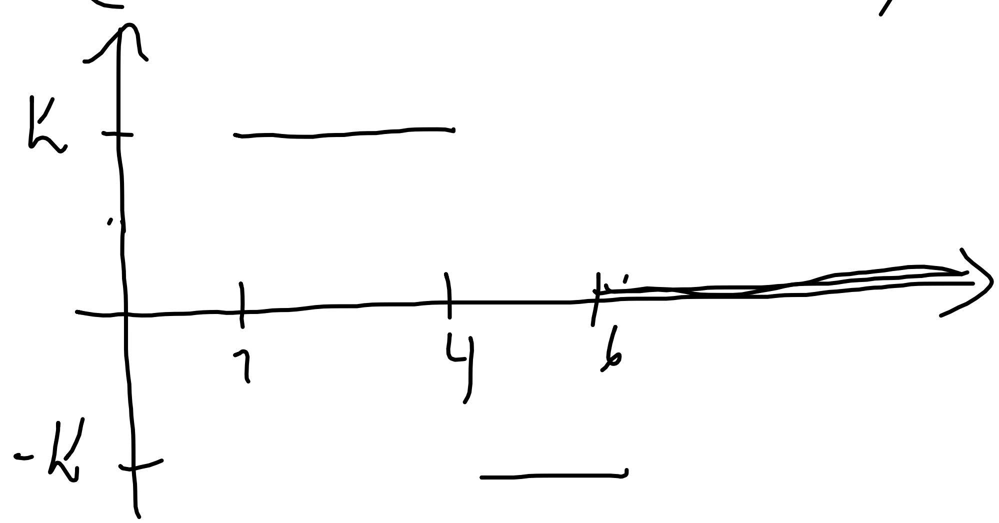
It is important to note

$$u(t-a) - u(t-b) = \begin{cases} 1 & \text{if } x \in (a,b) \\ 0 & \text{else} \end{cases}$$

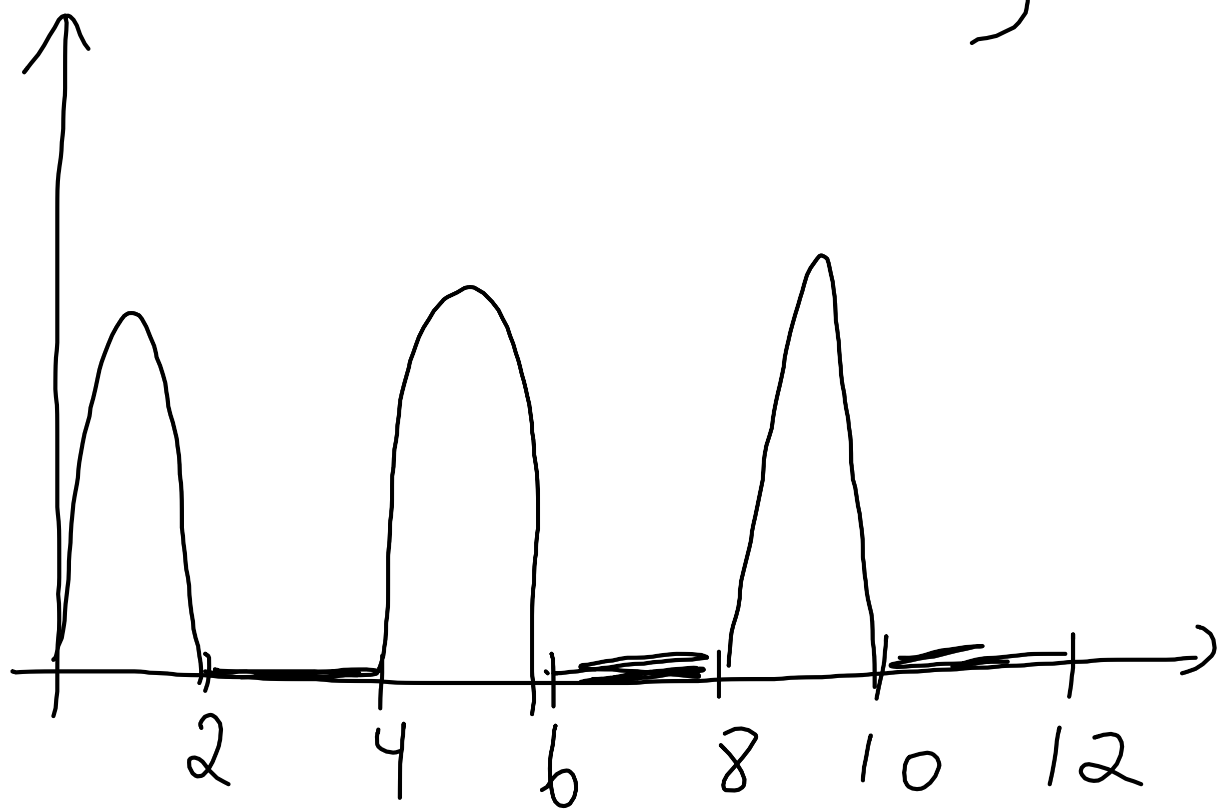
$b > a$



$$k\left[u(t-1)-2u(t-4)+u(t-6)\right]$$



$$4\sin\left(\frac{1}{2}\pi t\right)\left[\underbrace{u(t)}_{}^{\;1} - u(t-2) + u(t-4) - \cdots\right]$$



Second shifting theorem

The shifted function

$$\tilde{f}(t) = f(t-a)u(t-a)$$

$$= \begin{cases} 0 & \text{if } t<a \\ f(t-a) & \text{if } t>a \end{cases}$$

has LT

$$e^{-as}\mathcal{L}(f(t))(s)$$

Also

$$\mathcal{L}(f(t)u(t-a))$$

$$= e^{-as}\mathcal{L}(f(t+a))(s).$$

Example: Find the transform of

$$f(t) = \begin{cases} t^2/6 & \text{if } 0<t<1/2 \\ \sin 3t & \text{if } t>\pi/6 \\ 0 & \text{else} \end{cases}$$

$$= \frac{t^2}{6}\Big(\underbrace{u(t) - u(t-1/2)}_{\begin{cases}1 & 0<t<1/2\\ 0 & \text{else}\end{cases}}\Big)$$

$$+ \sin 3t \cdot u(t - \pi/6)$$

Applying LT and using 2^{nd} shifting theorem

$$\mathcal{L}\left(\frac{t^2}{6}\right) - \mathcal{L}\left(\frac{t^2}{6}u(t-1/2)\right)$$

$$+ \mathcal{L}\left(\sin 3t \; u(t - \pi/6)\right)$$

$$= \frac{1}{3s^3} - e^{-1/2 s}\mathcal{L}\left(\frac{(t+1/2)^2}{6}\right)$$

$$+ e^{-\pi/6 s}\mathcal{L}\left(\sin 3\left(t+\pi/6\right)\right)$$

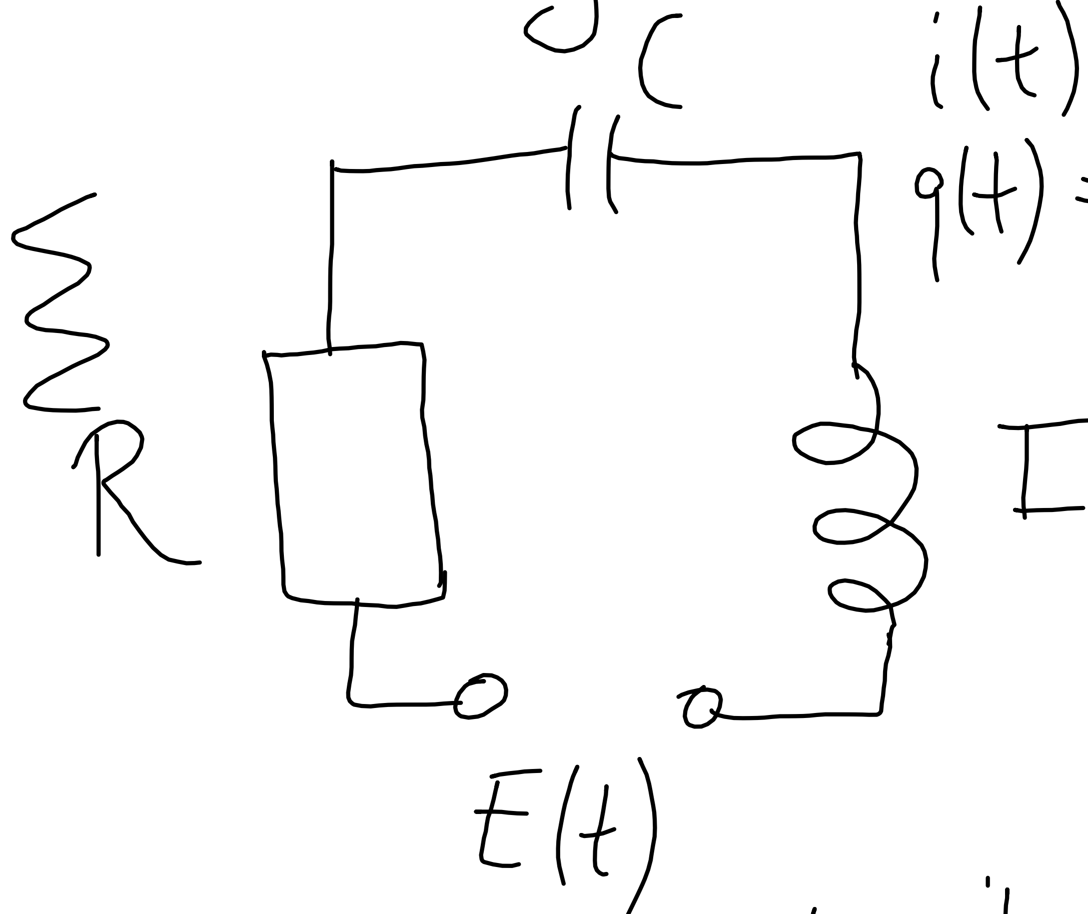
$$= \frac{1}{3s^3} - e^{-1/2 s}\mathcal{L}\left(\frac{t^2}{6} + \frac{t}{6} + \frac{1}{24}\right)$$

$$+ e^{-\pi/6 s}\mathcal{L}\left(\cos 3t\right) \quad \left(\sin(x+\pi/2) = \cos x\right)$$

$$= \frac{1}{3s^3} - e^{-1/2 s}\left(\frac{1}{3s^3} + \frac{1}{6s^2} + \frac{1}{24s}\right)$$

$$+ e^{-\pi/6 s}\left(\frac{s}{s^2+9}\right)$$

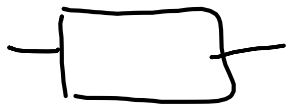
Modelling circuits

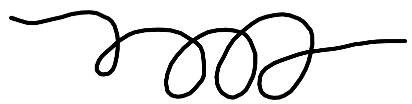


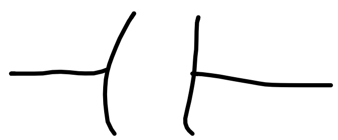
$i(t)$ = current
$q(t)$ = charge

		Unit	
R	resistor	Ω	ohms
I	inductor	H	henrys
C	capacitor	F	farads

Each induces a voltage drop

 RI

 $L\dfrac{di(t)}{dt}$

 $q(t)/C$

$$i(t) = \frac{dq(t)}{dt}$$

Kirchoff's Voltage law

$$Li'(t) + Ri(t) + \frac{q(i)}{C} = E(t)$$

$$Li'(t) + Ri(t) + \frac{1}{C}\int_0^t i(\alpha)\,d\alpha = E(t)$$

Taking derivaties

$$\boxed{Li''(t) + Ri'(t) + \frac{1}{C}i(t) = E(t)}$$