Fourier transforms

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)\, e^{-iv\omega}\, dv$$

$$\| \quad \mathcal{F}(f(x))(\omega)$$

$$\text{Diff. equation} \quad \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \quad \text{alg. equ. (simpler)}$$

Properties

$$f' \xrightarrow{\mathcal{F}} (i\omega)\mathcal{F}(f)$$

But the Fourier transform only exists if

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$$

(very strong)

e.g. e^x is not abs. integrable

1 is not abs. integrable

In particular, need

$$\lim_{x \to \pm\infty} f(x) = 0.$$

Heat equation (12.7)

Solve the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

for a bar of infinite length (very long)

There is no boundary condition (no boundaries)

Initial condition:

$$u(x,0) = f(x) \quad \text{for all } x \in \mathbb{R}$$

We can solve this equation using <u>separation of variables</u>

$$u(x,t) = F(x)G(t)$$

Get

$$\begin{cases} F'' + p^2 F = 0 \\ \dot{G} + c^2 p^2 G = 0 \end{cases}$$

$$(k = -p^2)$$

Solutions are of the form

$F_p(x) = A(p)\cos px + B(p)\sin px$

$G_p(t) = e^{-c^2p^2t}$

(No boundary conditions $\Rightarrow$ p is any number >0)

<u>Step 3</u>: In general, none of these particular solutions will solve for the initial conditions.

So we integrate over all p.

$$u(x,t) = \int_0^\infty [A(p)\cos px + B(p)\sin px]\, e^{-c^2p^2t}\, dp \qquad (1)$$

Using the initial condition:

$$f(x) = u(x,0) = \int_0^\infty [A(p)\cos px + B(p)\sin px]\, dp$$

(Fourier real integral for f)

$$\Rightarrow A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos pv\, dv$$

$$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin pv\, dv$$

② $$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_{-\infty}^{\infty} e^{-c^2p^2t}\cdot \cos(px-pv)\,dp\right]dv$$

③ $$u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)\, e^{-\frac{(x-v)^2}{4c^2t}}\,dv$$

④ $$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2cz\sqrt{t})\,e^{-z^2}\,dz$$

Use of Fourier transforms

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u(x,t)$

Apply the Fourier transform on both sides of the equation, thinking of u as a function of x (t is just a parameter)

$$\mathcal{F}(u(\underset{\text{variable}}{x},\underset{\text{number}}{t})) = \text{function of } \omega \;\&\; t.$$

$$\mathcal{F}(u_t) = c^2 \mathcal{F}(\underbrace{u_{xx}}_{\text{taking derivatives twice with respect to } x})$$

property 2

$$= c^2 (i\omega) \mathcal{F}(u_x)$$

$$= c^2 (i\omega)(i\omega) \mathcal{F}(u)$$

$$= -c^2 \omega^2 \mathcal{F}(u)$$

Since $\mathcal{F}$ thinks of u as a function of x

$$\mathcal{F}(u_t) \neq i\omega \mathcal{F}(u)$$

$$\mathcal{F}(u_t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u e^{-i\omega x}\, dx$$

$$= \frac{\partial}{\partial t} \mathcal{F}(u)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \mathcal{F}(u) = -c^2 w^2 \mathcal{F}(u)}$$

This is an ODE of $\mathcal{F}(u)_{(w,t)}$ with respect to t.

A general solution is of the form

$$\mathcal{F}(u)(w,t) = C(w)\, e^{-c^2 w^2 t}$$

$t = 0$ to find $C(w)$

$$C(w) = \mathcal{F}(u)(w, 0)$$

Recall: $u(x,0) = f(x)$

$$\Rightarrow \mathcal{F}(u)(w,0) = \mathcal{F}(f(x))(w)$$

$$= \hat{f}(w)$$

$$\Rightarrow C(w) = \hat{f}(w)$$

$$\Rightarrow \mathcal{F}(u)(w,t) = \hat{f}(w) e^{-c^2 w^2 t}$$

solution initial condition

We get u by applying Inverse FT on both sides

$$u(x,t) = \mathcal{F}^{-1}\left(\hat{f}(w)e^{-c^2w^2t}\right)$$

$$\boxed{= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{-c^2w^2t}e^{iwx}\,dw}$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{iwv}\,dv$$

Note: We have implicitly assumed that $\mathcal{F}(u)$ ~~exists~~ exists

$$\Rightarrow \int_{-\infty}^{\infty} |u|\,dx < \infty.$$

<u>Example</u>: Find a solution $u(x,t)$ to the heat equation with initial condition

$$\underset{f(x)}{\underset{\parallel}{u(x,0)}} = \begin{cases} |x| & \text{if } -1 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

We note that $u(x,0)$ is even. Applying the formula (Fourier integral)

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \int_0^1 v \cos pv \, dv \, (\cos px) e^{-c^2 p^2 t} \, dp$$

$B(p) = 0$ since f is even

Computing the First integral:

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\cos p + p\sin p - 1}{p^2} \cdot \cos px\, e^{-c^2p^2t}\,dp$$

Example:

Use the Fourier transform to solve the PDE

$$t u_x = u_t.$$

satisfying $u(x,0) = f(x)$.

We apply the FT on both sides (thinking of u as a function of x)

$$\mathcal{F}\big(\underbrace{t}_{\text{only a "number" in } \mathcal{F}} u_x(x,t)\big)(w,t) = \mathcal{F}\big(u_t(x,t)\big)(w,t)$$

$$\Rightarrow t\,\mathcal{F}(u_x) = \mathcal{F}(u_t)$$

$$\Rightarrow t\, i w\, \mathcal{F}(u) = \frac{\partial}{\partial t}\mathcal{F}(u)$$

This is an ODE of $\mathcal{F}(u)$ with respect to t.

A general solution is

$$\mathcal{F}(u)(w,t) = C(w)\, e^{t^2/2\, iw}$$

$$C(w) = \mathcal{F}(u)(w,0) = \mathcal{F}(u(x,0))$$
$$= \mathcal{F}(f(x))_{(w)}$$
$$= \hat{f}(w)$$

$$\Rightarrow \mathcal{F}(u) = \hat{f}(w)\, e^{t^2/2\, iw}$$

Apply the inverse FT.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{t^2/2\, iw} \cdot e^{iwx}\, dw$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{iw\left(\frac{t^2}{2} + x\right)}\, dw$$

This is the inverse FT at $\frac{t^2}{2} + x$

$$\boxed{= f\left(\frac{t^2}{2} + x\right)}$$

Laplace transform (chapter 6)

FT

PDE/ODE $\xrightarrow{\mathcal{F}}$ Get something nice $\xrightarrow{\mathcal{F}^{-1}}$ PDE/ODE

In order to apply $\mathcal{F}$, we assume that $\mathcal{F}(u)$ exists, which is not the case often.

Example: $y' = y$

We already know $y = Ce^t$

Say that we want to solve it using FT.

$$\mathcal{F}(y') = \mathcal{F}(y)$$

$$i\omega \mathcal{F}(y) = \mathcal{F}(y)$$

$$\Rightarrow \underbrace{(1 - i\omega)}_{\neq 0} \mathcal{F}(y) = 0$$

$$\Rightarrow \mathcal{F}(y) = 0$$

$\Rightarrow y = 0$

The reason why we missed the solution is because $\mathcal{F}(e^t)$ does not exist.

$$\int_{-\infty}^{\infty} e^t \, dt = \infty$$

$\Rightarrow$ FT method does not see this solution.

The goal is to modify FT to "see"

$$|F(x)| \le Me^{At}$$

<u>Definition</u>: If $f(t)$ $t \geq 0$

Its <u>Laplace transform</u> is a complex-valued function

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$s = \sigma + i\omega$$

$$= \int_0^\infty \underbrace{e^{-\sigma t}}_{\text{new}} \cdot \underbrace{e^{-i\omega t} f(t)\, dt}_{\text{had before with FT}}$$

this will cancel exponential growth in $f(t)$.

In this course, S will always be real.