Fourier integrals: $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\omega) e^{i\omega x} \, d\omega$

$$A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} \, dv$$

Example: $f(x) = \begin{cases} e^{-x} & ; \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

Use Fourier integrals to compute

$$\int_0^{\infty} \frac{\cos\omega + \omega \sin\omega}{1+\omega^2} \, d\omega \;.$$

$$A(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1+i\omega}$$

$$\underbrace{f(x)}_{\text{real}} = \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{1+i\omega} \cdot \frac{1-i\omega}{1-i\omega} \, d\omega}_{\text{complex}} = a + i\underbrace{b}_{=0}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(1-i\omega) e^{i\omega x}}{1+\omega^2} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(1-i\omega)(\cos\omega x + i \sin\omega x)}{1+\omega^2} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos\omega x - i\omega \cos\omega x + i \sin\omega x + \omega \sin\omega x}{1+\omega^2} \, d\omega$$

$$= \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos\omega x + \omega \sin\omega x}{1+\omega^2} \, d\omega}_{\text{real}} + \underbrace{\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{-\omega \cos\omega x + \sin\omega x}{1+\omega^2} \, d\omega}_{\substack{\text{imaginary} \\ =0}}$$

Making a clever choice for x $(x=1)$

$$f(1) = e^{-1} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\frac{\cos w + w\sin w}{1+w^2}}_{\text{even}} dw$$

$$= \frac{1}{\pi} \underbrace{\int_0^{\infty} \frac{\cos w + w\sin w}{1+w^2} dw}_{=\pi e^{-1}}$$

Fourier transforms:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(w) e^{iwx} dw$$

We call $A(w)$ the Fourier transform of $f(x)$.

↳ Depends on w.

Denoted by $\hat{f}(x)(w)$

$\mathcal{F}(f(x))(w)$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(x)(w) e^{iwx} dw$$

$$\hat{f}(x)(w) = 1/\sqrt{2\pi}\int_{-\infty}^{\infty} f(v) e^{-iwv} dv$$

Those two processes are inversed of each other, so we call the inverse Fourier transform denoted by $\mathcal{F}^{-1}$

$$\mathcal{F}^{-1}(g(\omega))(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{i\omega x}\,d\omega$$

$$\mathcal{F}(\mathcal{F}^{-1}(g)) = g \qquad \mathcal{F}^{-1}(\mathcal{F}(f)) = f$$

Intuitively, the function f is the sum of all sinusoidal waves whose amplitudes are given by the Fourier transform.

Inversely, the Fourier transform gives the amplitude (weight) of the sinusoidal waves of frequency ω inside f.

Existence of the Fourier transform

If $f(x)$ is absolutely integrable, i.e

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$$ and piecewise continuous

strong

then $\hat{f}(\omega)$ exists.

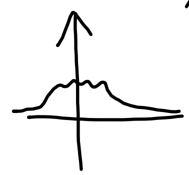
Properties: 1) Linearity of FT.

If f & g are two functions and a & b are numbers

$$\Rightarrow \mathcal{F}(af+bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

(because $\mathcal{F}$ is an integral)

2) If f is continuous, f' absolutely integrable and $\underbrace{\lim_{x\to\pm\infty} f(x) = 0}_{\text{automatic if } f \text{ is absolutely integrable}}$



$$\boxed{\mathcal{F}(f')(\omega) = +i\omega\mathcal{F}(f)(\omega)}$$

3) $$\boxed{\mathcal{F}(xf(x))(\omega) = -i\mathcal{F}'(f(x))(\omega)}$$

Proof of 2): $\mathcal{F}(f')(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{f'(v)}_{d\tilde{v}} \underbrace{e^{-i\omega v}}_{u} dv$

Use integration by parts

$$= \frac{1}{\sqrt{2\pi}}\left(\underbrace{\Big[\underbrace{f(v)}_{\to 0 \text{ as } v\to\pm\infty} \underbrace{e^{-i\omega v}}_{\text{bounded}}\Big]_{-\infty}^{\infty}}_{=0} - (-i\omega)\int_{-\infty}^{\infty} f(v) e^{-i\omega v} dv \right)$$

$$= i\omega \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v) e^{-i\omega v} dv = i\omega\mathcal{F}(f).$$

In general, $\mathcal{F}\left(f^{(k)}\right) = (i\omega)^k \mathcal{F}(f)$

<u>Convolution</u>

The convolution of two functions f, g is defined as

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt = \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$$

<u>Convolution theorem</u>: Same hypothesis as usual:

$$\mathcal{F}(f * g) = \sqrt{2\pi}\, \mathcal{F}(f) \cdot \mathcal{F}(g)$$

This implies

$$(f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)\, \hat{g}(\omega)\, e^{i\omega x}\, d\omega$$

(Applying the inverse FT on both sides)

Convolution $\rightsquigarrow$ multiplication of FT.

<u>Example</u>: Find the function $f(x)$ s.t.

$$\int_{-\infty}^{\infty} f(x-t)\, e^{-2t^2}\, dt = e^{-x^2} \quad \text{for all } x \in \mathbb{R}.$$

$$\overset{\shortparallel}{(f * g)(x)} \quad \text{where } g(x) = e^{-2x^2}.$$

Equation becomes

$$f * e^{-2x^2} = e^{-x^2}$$

We apply FT on both sides

$$\mathcal{F}\left(f * e^{-2x^2}\right) = e^{-x^2}$$

Use convolution theorem:

$$\sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}\left(e^{-2x^2}\right) = \mathcal{F}\left(e^{-x^2}\right)$$

$$\Longrightarrow \quad \mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \cdot \frac{\mathcal{F}\left(e^{-x^2}\right)}{\mathcal{F}\left(e^{-2x^2}\right)}$$

We can find F by applying the inverse FT on both side.

But first, we need to compute $\mathcal{F}(e^{-2x^2})$ & $\mathcal{F}(e^{-x^2})$.

We will compute $\mathcal{F}\Big(\overbrace{e^{-b^2x^2}}^{h(x)}\Big)$ $b \neq 0$.

$\lim_{x\to\pm\infty} e^{-b^2x^2} = 0$

so we can apply properties of FT.

$$h'(x) = -2b^2xe^{-b^2x^2} = -2b^2xh(x)$$

Applying FT on both sides

$$\mathcal{F}(h'(x)) = -2b^2\mathcal{F}(xh(x))$$

Use properties ② ③

$$i\omega\mathcal{F}(h(x))(\omega) = -2b^2 i\,\mathcal{F}'(h(x))(\omega)$$

This is an ODE for the function $\mathcal{F}(h(x))(\omega)$ which depends on ω.

$$\Rightarrow \mathcal{F}(h)(\omega) = Ce^{-\frac{\omega^2}{4b^2}}$$

$$C = \mathcal{F}(h)(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-b^2v^2}\underbrace{e^{i\cdot 0\cdot v}}_{=1}\,dv \quad (\omega = 0)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-b^2v^2}\,dv}_{\text{Gaussian integral}} = \frac{1}{\sqrt{2}\,b}$$

$$\Rightarrow \underset{\mathcal{F}(e^{-b^2x^2})(\omega)}{\mathcal{F}(h)(\omega)} = \frac{1}{\sqrt{2}\,b}e^{-\omega^2/4b^2}$$

Going back to the original equation:

$$\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \cdot \frac{\mathcal{F}(e^{-x^2})}{\mathcal{F}(e^{-2x^2})} = \frac{1}{\sqrt{2\pi}} \frac{\frac{1}{\sqrt{2}} e^{-w^2/4}}{\frac{1}{2} e^{-w^2/8}}$$

$$= \frac{1}{\sqrt{\pi}} e^{-w^2/8}$$

We can apply the inverse FT

$$f = \mathcal{F}^{-1}\Big(\underbrace{1/\sqrt{\pi}\, e^{-w^2/8}}\Big)$$

We already know that this is the Fourier transform of e^{-x^2} (up to some numbers)

$$= 1/\sqrt{\pi} \cdot 2 \cdot \mathcal{F}^{-1}\left(1/2\, e^{-w^2/8}\right)$$

$$\boxed{= \frac{2}{\sqrt{\pi}} e^{-2x^2}}$$

Strategy: 1) Instead of solving the equation directly ⇝ Apply FT, compute there, & then apply the inverse FT.

2) For $\mathcal{F}(e^{-b^2x^2})$, used the fact that it satisfies a very nice ODE, and applied FT to it, using properties.

Fourier transform of 1

We found that

$$\mathcal{F}\left(e^{-b^2x^2}\right)(\omega) = \frac{1}{\sqrt{2}\,b}\, e^{-\omega^2/4b^2}$$

Take $\lim_{b\to 0}$ on both sides

$$\mathcal{F}\Big(\underbrace{1}_{\substack{\text{not}\\ \text{absolutely}\\ \text{integrable}}}\Big) = \lim_{b\to 0} \frac{1}{\sqrt{2}b} e^{-\omega^2/4b^2} = \delta$$

δ = Dirac delta function (it is not a function)

$$= \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{else} \end{cases}$$