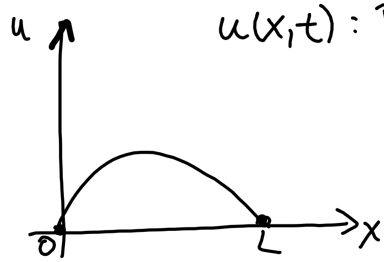

$u(x,t)$: Deflection of the string at point x, time t

$$\left.\begin{aligned} u(0,t)&=0 \\ u(L,t)&=0 \end{aligned}\right\}\ \text{Boundary conditions}$$

$$\left.\begin{aligned} u(x,0)&=f(x) \\ u_t(x,0)&=g(x) \end{aligned}\right\}\ \text{Initial conditions}$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

Step 1: Find all solutions of the form $u(x,t)=F(x)G(t)$ (separation)

$$\Longrightarrow \begin{cases} F'' - kF = 0 \\ \ddot{G} - c^2 k G = 0 \end{cases}$$

Step 2: Find solutions that satisfy the boundary conditions

$u(0,t)=0$ & $u(L,t)=0$ for all $t \geq 0$

$$\underbrace{F(0)}_{①=0}\,\underbrace{G(t)}_{\cancel{②}\neq 0 \text{ for all } t} \qquad \underbrace{F(L)}_{=0}\,\underbrace{G(t)}_{\neq 0 \text{ for all } t}$$

$$\boxed{F(0)=0 \qquad F(L)=0}$$

Solve $F'' - kF = 0$

① $k=0$: $F''=0 \Longrightarrow F=ax+b$

$$\left.\begin{aligned} 0&=F(0)=b \Rightarrow b=0 \\ 0&=F(L)=aL \Rightarrow a=0 \end{aligned}\right\}\ F=0 \text{ for all } x$$

② $k>0$: General solution $(k=\lambda^2)$

$$F = Ae^{\mu_1 x} + Be^{\mu_2 x}$$

where μ_1 & μ_2 are solutions

$$\mu^2 - k = 0$$

$$\Rightarrow \mu = \pm\lambda$$

$$\Rightarrow F = Ae^{\lambda x} + Be^{-\lambda x}$$

$$\left.\begin{array}{l} 0 = F(0) = A + B \Rightarrow A = -B \\ 0 = F(L) = A\underbrace{e^{\lambda L}}_{>0} - A\underbrace{e^{-\lambda L}}_{>0} \end{array}\right\} A = B = 0$$

If $k<0$ $(k=-p^2)$ $(\mu^2+p^2=0)$
the general solution

$$F(x) = A\cos px + B\sin px$$

$$0 = F(0) = A$$

$$0 = F(L) = B\sin(Lp) \Rightarrow pL = n\pi \text{ for some integer } n.$$

Solutions: $\boxed{F_n(x) = \cancel{B}^{\,1} \sin\left(\frac{n\pi}{L}x\right)}$

Solve $\ddot{G} - c^2kG = 0$ $(k=-p^2)$ $\left(p = \frac{n\pi}{L}\right)$

$$\Rightarrow \ddot{G} + (cp)^2 G = 0$$

General solution:

$$\boxed{\begin{array}{l} G_n(t) = B_n\cos(\lambda_n t) + B_n^*\sin(\lambda_n t) \\ \lambda_n = cp = c\cdot\frac{n\pi}{L} \end{array}}$$

General solution for Wave equation:

$$\boxed{\begin{aligned} u_n(x,t) &= F_n(x)\,G_n(t) \\ &= \left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t\right) \sin \frac{n\pi}{L} x \end{aligned}}$$

These are called Eigenfunctions (characteristic fcts)

λ_n : Eigenvalues

$\{\lambda_1, \lambda_2, \lambda_3, \dots\}$: Spectrum

Each u_n represents a harmonic motion having frequency $\boxed{\dfrac{\lambda_n}{2\pi}}$

This motion is called the n^{th} normal mode of the string. $u_1(x,t)$ is called Fundamental mode

Since $\overset{F_n}{\overset{\shortparallel}{\sin \frac{n\pi x}{L}}} = 0$

for $x = \frac{L}{n}, \frac{2L}{n}, \frac{3L}{n}, \dots, \frac{n-1}{n} L$

$\Longrightarrow$ n^{th} normal mode has $(n-1)$ nodes $\rightsquigarrow$ string does not move.

Step 3: Want a general solution that satisfies the initial conditions.

However, in general, none of the $u_n(x,t)$ will.

The strategy is to take infinite sum:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t \right) \sin \frac{n\pi}{L} x$$

Want it to satisfy the initial conditions

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} \left(B_n \sin \frac{n\pi}{L} x \right)$$

Fourier sine series of the odd expansion of $f(x)$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \quad n = 1, 2, 3, \dots$$

The other initial condition

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t \right) \sin \frac{n\pi}{L} x$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(B_n \cdot -\lambda_n \sin\lambda_n t + B_n^* \lambda_n \cos\lambda_n t \right) \sin \frac{n\pi}{L} x$$

$$g(x) = \left. \frac{\partial u}{\partial t} \right|_{t=0} = \sum_{n=1}^{\infty} \underline{B_n^* \lambda_n} \sin \frac{n\pi}{L} x$$

Fourier sine series of odd expansion of $g(x)$

$$\boxed{B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx \quad n = 1, 2, 3, \dots}$$

If $g(x) = 0$ (no initial velocity)

$$\Rightarrow B_n^* = 0$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos \lambda_n t \sin \frac{n\pi x}{L} \right)$$

Trigo. identity:

$$\cos \frac{cn\pi}{L} t \sin \frac{n\pi}{L} x = \frac{1}{2}\left[\sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \sin\left\{\frac{n\pi}{L}(x+ct)\right\}\right]$$

$$\Rightarrow u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \left(\sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \sin\left\{\frac{n\pi}{L}(x+ct)\right\}\right)$$

Fourier sine series $\leftarrow$

$$= \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct) \right]$$

$\hookrightarrow$ odd expansion of f.