Partial differential equations

Def: A PDE is an equation involving partial derivatives

Examples:

lin. homog. ← 1. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (order 2)

lin. non-homog. ← 2. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ (order 2)

non-linear homogen. ← 3. $\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \frac{\partial u}{\partial x} = \left(\frac{\partial^3 u}{\partial x^2 \partial y}\right)^5$ (order 3)

Def The order of a PDE is the order of its highest derivative.

Def: We say that a PDE is linear if each of its partial derivative is in degree 1. It is homogeneous if each of its terms contain a partial derivative of the function itself.

Notation: We will often denote

$$\frac{\partial^2 u}{\partial x \partial y} = u_{xy}$$

Definition: A solution of a PDE in some region R is a function that satisfies the PDE in R.

In general, the totality of solutions to a PDE is very <u>Large</u>.

Laplace equation: $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

has solutions:

$u=x^2-y^2$, $u=e^x\cos y$, $u=\sin x\cosh y$

$u=\ln(x^2+y^2)$

In order to get a unique solution, we impose additional conditions, coming from the context. These could be values for u on the boundary of R (Boundary conditions) or initial values for u (initial conditions).

As it it the case for ODEs, we have the theorem on superposition for linear homog. equations:

<u>Thm</u>: If u_1 & u_2 are solutions of a homog. lin. PDE in some region R then $u = C_1 u_1 + C_2 u_2$ is also a solution of that PDE in R f. any $c_1, c_2 \in \mathbb{R}$.

PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} = 0$ two solutions u_1 & u_2

$$\frac{\partial^2 (c_1 u_1 + c_2 u_2)}{\partial x^2} + \frac{\partial (c_1 u_1 + c_2 u_2)}{\partial y} = c_1 \frac{\partial^2 u_1}{\partial x^2} + c_2 \frac{\partial^2 u_2}{\partial x^2} + c_1 \frac{\partial u_1}{\partial y} + c_2 \frac{\partial u_2}{\partial y} = 0$$

Examples: Solve for $u = u(x,y)$, the equation

$$25u_{yy} - 4u = 0$$

Since the equation involves only partial derivatives with respect to one variable (y), we can think of it as an ODE

$$25u'' - 4u = 0$$

We know that the solutions are of the form

$$u(y) = \underline{A}e^{\lambda_1 y} + \underline{B}e^{\lambda_2 y}$$

where λ_1 & λ_2 are solutions

$$25\lambda^2 - 4\lambda = 0 \qquad \lambda_1 = \frac{2}{5}, \lambda_2 = -\frac{2}{5}$$

Since $u = u(x,y)$, get

$$u(x,y) = A(x)e^{2/5y} + B(x)e^{-2/5y}$$

where A & B are arbitrary functions of x.

Example: Solve $u = u(x,y)$

$$u_{xy} = u_x \quad \star$$

Define $f(x,y) = u_x$

$$\Rightarrow u_{xy} = f_y(x,y)$$

Then the equation becomes

$$f_y = f. \quad \frac{\partial f}{\partial y} = f$$

which can be treated as an ODE.

$$\int \frac{\partial f}{f} = \int \partial y$$

$$\Longrightarrow \ln|f| = y + \tilde{c}(x)$$

$$\Longrightarrow u_x = f = C(x)e^y$$

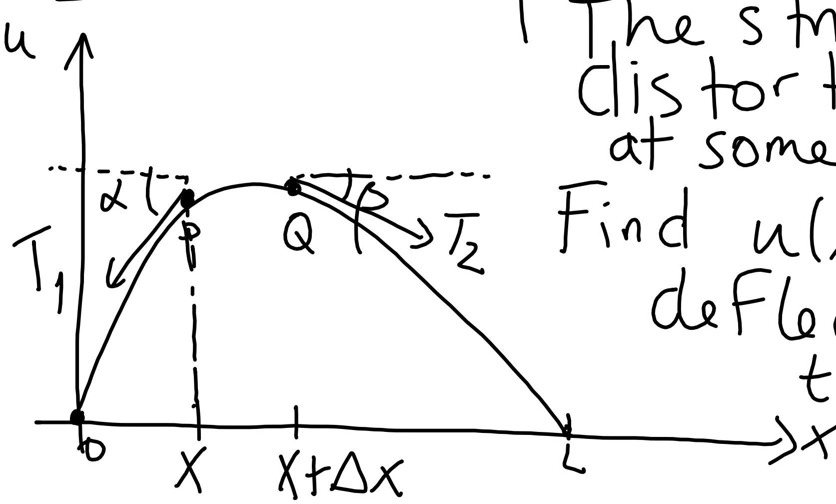
Integrate with respect to x :

$$u(x,y) = A(x)e^{y} + B(y)$$

where $A(x) = \int c(x)\,dx$

A & B are arbitrary functions

Modeling: Vibrating string & Wave equation



The string is distorted an released at some time $t=0$

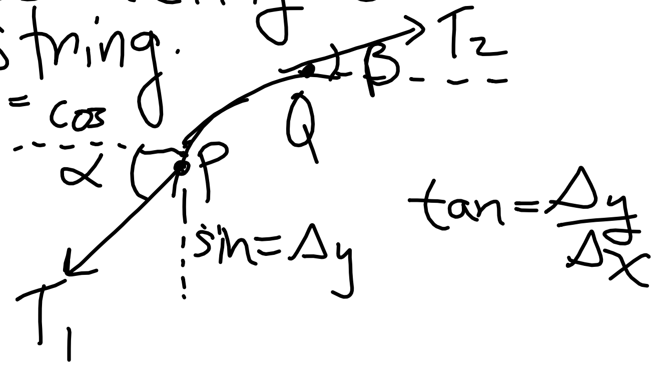
Find $u(x,t)$, the deflection of the string at any point x

time $t>0$.

Assumptions

1. The mass density is constant.
2. The initial tension in the string is so large that we can neglect the action of gravity.
3. The string only moves vertically.

To obtain a PDE, we consider the forces acting on a small portion of the string.



No horizontal motion $\Longrightarrow$

(1) $T_1 \cos\alpha - T_2 \cos\beta = 0$ (horizontally)

$\Longrightarrow T_1 \cos\alpha = T_2 \cos\beta = T$

Vertically: Newton's second law

(2) $$T_2 \sin\beta - T_1 \sin\alpha = \underbrace{\rho}_{\text{mass density}} \Delta x \frac{\partial^2 u}{\partial t^2}$$

(2)/(1): $$\frac{\cancel{T_2} \sin\beta}{\cancel{T_2} \cos\beta} - \frac{\cancel{T_1} \sin\alpha}{\cancel{T_1} \cos\alpha} = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$\Longrightarrow$ (3) $$\tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

Get $\tan\alpha = \left(\frac{\partial u}{\partial x}\right)\Big|_{x}$ $\tan\beta = \left(\frac{\partial u}{\partial x}\right)\Big|_{x+\Delta x}$

(slopes of tangent lines at P & Q)

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)\Big|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)\Big|_{x}\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

Taking $\lim\limits_{\Delta x \to 0}$:

$$\frac{\partial^2 u}{\partial x^2} = \underbrace{\frac{\rho}{T}}_{= 1/c^2}\frac{\partial u^2}{\partial t^2}$$

$$\Longrightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$ Wave equation.

In order to be able to solve this equation, need additional conditions. $R = \underbrace{[0, L]}_{x} \times \underbrace{(0, \infty)}_{t}$

$u(0,t) = 0 \qquad u(L,t) = 0$

for all $t > 0$

(Boundary conditions)

$\underbrace{u(x,0) = f(x)}_{\text{Initial position}} \qquad \underbrace{u_t(x,0) = g(x)}_{\text{Initial velocity}}$

for all $x \in [0, L]$

(Initial conditions)