Laplace transforms (chap 6)

The goal is to get a transform that exists for $f(t)$ such that

$$|f(t)| \leq M e^{kt}$$

for some M & k.

For example, NOT

$$e^{t^2}.$$

Definition: (Laplace transform)

If $f(t)$ is defined for all $t \geq 0$, its **Laplace transform** is a complex-valued function

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\, dt$$

where s is a complex number.

$$s = \sigma + i\omega$$

$$\mathcal{L}(f) = \int_0^\infty e^{-\sigma t} \cdot \underbrace{e^{-i\omega t} f(t)\, dt}_{\text{Original FT}}$$

Compensate for exponential growth.

In this course, we only consider s real.

The function $f(t)$ is the inverse Laplace transform. It is denoted by $\mathcal{L}^{-1}$. of $F(s)$.

Example: LT of $f(t) = 1$.

$$\mathcal{L}(1) = \int_0^\infty 1 \cdot e^{-st}\,dt$$

$$= -\frac{1}{s} e^{-st}\Big]_0^\infty = \frac{1}{s}$$

So the inverse LT of $F(s)=\frac{1}{s}$ is $f(t)=1$.

<u>Example</u>: $f(t)=e^{at}$, $t \geq 0$

$$\mathcal{L}(e^{at})(s)=\int_0^\infty e^{-st} \cdot e^{at}\,dt$$

$$=\int_0^\infty e^{-(s-a)t}\,dt$$

$$=-\frac{1}{s-a}e^{-(s-a)t}\Big]_0^\infty = \frac{1}{s-a}$$

if $s>a$.

Theorem:

If $|f(t)| \leq M e^{kt}$
for some M & k and
f is piecewise continuous.
on every finite interval.
Then
$\mathcal{L}(f)$ exists for $s > k$. and
it determines uniquely f.

Properties:

1. Linearity of the LT

$$\mathcal{L}(af(t)+bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$$

$a, b \in \mathbb{R}$

Example: $\mathcal{L}(\cosh at)$

$$= \frac{1}{2}\left(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2}$$

2. LT of derivatives

$$\mathcal{L}(f^{(n)}(t)) = s^n \mathcal{L}(f(t))$$

$$- s^{n-1} f(0) - s^{n-2} f'(0)$$

$$- \dots - f^{(n-1)}(0).$$

Example: $f(t) = \sin(\omega t)$

$$f'(t) = \omega \cos(\omega t)$$

$$f''(t) = -\omega^2 \sin(\omega t) = -\omega^2 f(t)$$

$$f(0) = 0 \quad f'(0) = \omega$$

$$f''(t) = -\omega^2 f(t)$$

$$\mathcal{L}(f''(t)) = -\omega^2 \mathcal{L}(f(t))$$

$$\shortparallel$$

$$s^2 \mathcal{L}(f) - s \cdot 0 - \omega = -\omega^2 \mathcal{L}(f)$$

$$\Rightarrow s^2 \mathcal{L}(f) - \omega = -\omega^2 \mathcal{L}(f)$$

$$\Rightarrow \boxed{\mathcal{L}(f) = \frac{\omega}{s^2 + \omega^2}}$$

3. LT of integrals

$$\mathcal{L}\left(\int_0^t f(\alpha)\,d\alpha\right) = \frac{1}{s}\mathcal{L}(f).$$

Example: Find the inverse LT of

$$F(s) = \frac{2}{s^2 + s/3}$$

$$= \frac{1}{s} \cdot \frac{2}{s+1/3}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s} \cdot \frac{2}{s+1/3}\right) = 2\int_0^t \mathcal{L}^{-1}\left(\frac{1}{s+1/3}\right)(\alpha)\, d\alpha$$

$$= 2\int_0^t e^{-1/3\alpha}\, d\alpha$$

$$= \boxed{6 - 6e^{-1/3t}}$$

4. Shifting theorem

$$\mathcal{L}(f)(s-a) = \mathcal{L}\left(e^{at} f(t)\right)(s)$$

Proof:

$$\mathcal{L}(f)(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= \int_0^\infty e^{-st} \cdot \underbrace{\left[e^{at} f(t)\right]}_{g(t)} dt$$

$$= \mathcal{L}(g(t))(s) = \mathcal{L}\left(e^{at} f(t)\right)(s)$$

Example:

Facts: $\mathcal{L}(\sin \omega t) = \dfrac{\omega}{s^2+\omega^2}$

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$$

Using this and shifting

$$\mathcal{L}(e^{at}\cos \omega t) = \frac{s-a}{(s-a)^2+\omega^2} \quad *$$

$$\mathcal{L}(e^{at}\sin \omega t) = \frac{\omega}{(s-a)^2+\omega^2}$$

Find the inverse LT of

$$F(s) = \frac{3s-137}{s^2+2s+401}$$

Rewriting:

$$\frac{3s-137}{s^2+2s+401} = \frac{3s-137}{s^2+2s+1+400}$$

complete the square

$$= \frac{3s-137}{(s+1)^2+20^2} = \frac{3(s+1)-140}{(s+1)^2+20^2}$$

$$= 3\frac{s+1}{(s+1)^2+20^2} - 7\frac{20}{(s+1)^2+20^2}$$

Applying inverse LT

$$\mathcal{L}^{-1}(F(s)) = 3e^{-t}\cos 20t - 7e^{-t}\sin 20t$$

Solving ODEs using LT

(6.2)

Consider the IVP

$$y'' + ay' + by = r(t) \quad (*)$$

$$y(0) = K_0 \quad y'(0) = K_1$$

a, b constants.

$r(t)$: input (driving force)

$y(t)$: output (response)

$\hookrightarrow$ time

$$\begin{tikzcd} \boxed{\text{ODE}} \arrow[r, bend left, "\mathcal{L}"] & \text{Algebraic equation} \arrow[d, squiggly] \\ & \text{rational function} \arrow[ul, bend left, "\mathcal{L}^{-1}"] \end{tikzcd}$$

$\mathcal{L}^{-1} \rightsquigarrow$ partial fractions, completing the square, etc

(use tables p. 249-251)

Apply $\mathcal{L}$ to $(\ast)$.

$$\mathcal{L}(y'' + ay' + by)(s) = \mathcal{L}(r)(s)$$

$\parallel$ LT for derivatives

$$[s^2\mathcal{L}(y) - sy(0) - y'(0)]$$

$$+a[s\mathcal{L}(y) - y(0)]$$

$$+b\mathcal{L}(y) \qquad = \mathcal{L}(r)$$

Isolating $\mathcal{L}(y)$, we obtain

$$\mathcal{L}(y) = \frac{(s+a)\,y(0) + y'(0)}{s^2 + as + b} + \frac{\mathcal{L}(r)}{s^2 + as + b}$$

(rational function of s)

Get y by applying the inverse LT.

Two advantages

1. You don't need to first find the homogeneous solution and then a particular solution.
2. The initial conditions are taken care of automatically.

We call

$$Q(s) = \frac{1}{s^2 + as + b}$$ the <u>transfer function.</u>

If $y(0) = 0$ and $y'(0) = 0$

we have

that $Q = \dfrac{\mathcal{L}(y)}{\mathcal{L}(r)} = \dfrac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$

It only depends on

a & b, not on r.