NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4130 Matematikk 4N**

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Examination time (from–to): 09:00–13:00

Permitted examination support material: Kode C:
Bestemt, enkel kalkulator
Rottmann: Matematisk formelsamling

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- All sub-problems carry the same weight for grading.
- Good Luck!

Language: English

Number of pages: 3

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Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
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skal ha flervalgskjema ☒	

Date Signature

Problem 1 Let $f(x)$ be defined as

$$f(x) = \frac{\pi}{4} - \frac{x}{2}, \qquad \text{where } 0 < x < \pi.$$

a) Find the Fourier cosine series of $f(x)$.

b) Use the result to compute the value of the series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

Problem 2 Solve the integral equation

$$\int_{-\infty}^{\infty} f(x-t)e^{-|t|}\,\mathrm{d}t = e^{\frac{-x^2}{2}}$$

using Fourier transforms. (**Hint**: You may need the formula for the Fourier transform of derivatives.)

Problem 3 Consider the ordinary differential equation

$$\begin{aligned} y'' - 3y' + 2y &= r, \\ y(0) &= 1, \\ y'(0) &= A \end{aligned} \tag{1}$$

where $r = r(t)$ is a given function and A is a constant.

a) Solve this equation using the Laplace transform in the case $r(t) = 0$.

b) Determine the solution in the case when $r(t) = e^t$.

c) Rewrite the equation (1) for an arbitrary $r = r(t)$ as a first-order system of the form

$$Z'(t) = F(t, Z)$$

where $F(t, Z)$ is a vector-valued function and $Z(t)$ is the unknown vector-valued function to be determined.

d) Write down the classical Runge–Kutta method for this system of equations.

e) Compute the approximate solution $y(0.1)$ using $h = 0.1$ in the case $r(t) = 0$ and $A = 0$. Keep 6 digits in the computations.

Problem 4 Consider the equation $e^{\frac{x}{3}} - x = 0$.

a) Show that this equation has a unique solution in the interval $(0, 3)$.

b) Compute 3 iterations of Newton's method to approximate the solution, starting with $x_0 = 1$.

c) Write the equation as an equation of the form $g(x) = x$ so that the fixed-point iteration method converges, and compute 3 iterations, starting with $x_0 = 1$.

Keep 5 digits in your computations.

Problem 5 Consider the function $f(x) = \ln(x)$.

a) Compute the Chebyshev points, with $n = 2$, in the interval $[0, 2]$.

b) Using Lagrange interpolation, find the polynomial of smallest degree that interpolates the function at the Chebyshev points found in **a)**.

Keep 5 digits in the computations.

Problem 6 Consider the system of equations

$$\begin{array}{rcrcrcr} 4x_1 & - & x_2 & + & 2x_3 & = & 20 \\ & - & x_2 & + & 4x_3 & = & 28 \\ -x_1 & + & 4x_2 & - & x_3 & = & -40 \end{array}$$

a) Rearrange this system of equations so that you can apply Jacobi's method and such that it converges.

b) Perform 2 iterations of the Jacobi's method, starting with $\mathbf{x}_0 = (1,\ 1,\ 1)$, using 5 digits in the computations.

Problem 7 Let $u(x,t)$ be the temperature at time t in a laterally insulated bar of length 3 lying on the x-axis. It satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le 3, \quad t \ge 0,$$

with initial condition

$$u(x,0) = 2\sin\left(\frac{\pi x}{3}\right)$$

and boundary conditions

$$u(0,t) = u(3,t) = 0, \qquad t \ge 0.$$

a) Find the solutions that are of the form $u(x,t) = F(x)G(t)$ and that satisfy the boundary conditions.

b) Find the solution that satisfies the initial condition. Evaluate $u(1, 0.5)$.

c) Use the Crank–Nicolson method with $k = 0.5$ and $h = 1$ to approximate the value of $u(1, 0.5)$. Keep 5 digits in the computations.

Fourier Transform

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\ \mathrm{d}\omega$	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\ \mathrm{d}x$
$f * g(x)$	$\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$
$f'(x)$	$i\omega\hat{f}(\omega)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$
$e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}} \frac{a}{\omega^2 + a^2}$
$\frac{1}{x^2 + a^2}$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\|\omega\|}}{a}$
$f(x) = 1$ for $\|x\| < a$, 0 otherwise	$\sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}$

Laplace Transform

$f(t)$	$F(s) = \int_0^{\infty} e^{-st} f(t)\ \mathrm{d}t$
$f'(t)$	$sF(s) - f(0)$
$tf(t)$	$-F'(s)$
$e^{at} f(t)$	$F(s - a)$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s - a}$
$f(t - a)u(t - a)$	$e^{-sa} F(s)$
$\delta(t - a)$	e^{-as}
$f * g(t)$	$F(s)G(s)$

Numerics

- Newton's method: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$.
- Newton's method for system of equations: $\vec{x}_{k+1} = \vec{x}_k - JF(\vec{x}_k)^{-1}F(\vec{x}_k)$, with $JF = (\partial_j f_i)$.
- Lagrange interpolation: $p_n(x) = \sum_{k=0}^{n} \frac{l_k(x)}{l_k(x_k)} f_k$, with $l_k(x) = \prod_{j\neq k}(x - x_j)$.
- Interpolation error: $\epsilon_n(x) = \prod_{k=0}^{n}(x - x_k)\frac{f^{(n+1)}(t)}{(n+1)!}$.
- Chebyshev points: $x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right)$, $0 \leq k \leq n$.
- Newton's divided difference: $f(x) \approx f_0 + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})f[x_0, \ldots, x_n]$, with $f[x_0, \ldots, x_k] = \frac{f[x_1,\ldots x_k] - f[x_0,\ldots,x_{k-1}]}{x_k - x_0}$.
- Trapezoid rule: $\int_a^b f(x)\,\mathrm{d}x \approx h\left[\frac{1}{2}f(a) + f_1 + f_2 + \cdots + f_{n-1} + \frac{1}{2}f(b)\right]$.
  Error of the trapezoid rule: $|\epsilon| \leq \frac{b-a}{12}h^2 \max_{x\in[a,b]} |f''(x)|$.
- Simpson rule: $\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n\right]$.
  Error of the Simpson rule: $|\epsilon| \leq \frac{b-a}{180}h^4 \max_{x\in[a,b]} |f^{(4)}(x)|$.
- Gauss–Seidel iteration: $\mathbf{x}^{(m+1)} = \mathbf{b} - \mathbf{L}\mathbf{x}^{(m+1)} - \mathbf{U}\mathbf{x}^{(m)}$, with $\mathbf{A} = \mathbf{I} + \mathbf{L} + \mathbf{U}$.
- Jacobi iteration: $\mathbf{x}^{(m+1)} = \mathbf{b} + (\mathbf{I} - \mathbf{A})\mathbf{x}^{(m)}$.
- Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$.
- Improved Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(x_n, \mathbf{y}_n) + \mathbf{f}(x_n + h, \mathbf{y}^*_{n+1})]$, where $\mathbf{y}^*_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$.
- Classical Runge–Kutta method: $\mathbf{k}_1 = h\mathbf{f}(x_n, \mathbf{y}_n)$,
  $\mathbf{k}_2 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_1/2)$, $\mathbf{k}_3 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_2/2)$,
  $\mathbf{k}_4 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_3)$, $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{6}\mathbf{k}_4$.
- Backward Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_{n+1}, \mathbf{y}_{n+1})$.
- Finite differences: $\frac{\partial u}{\partial x}(x, y) \approx \frac{u(x+h,y) - u(x-h,y)}{2h}$, $\frac{\partial^2 u}{\partial x^2}(x, y) \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$.
- Crank–Nicolson method for the heat equation: $r = \frac{k}{h^2}$,
  $(2 + 2r)u_{i,j+1} - r(u_{i+1,j+1} + u_{i-1,j+1}) = (2 - 2r)u_{ij} + r(u_{i+1,j} + u_{i-1,j})$.