Norwegian University of Science and Technology
Department of Mathematical Sciences

Chebyshev points

The interpolation error depends on the choice of points on $f(x)$. Recall that, if $f^{(n+1)}$ exists and is continuous, the error is given by

$$\epsilon_n(x) = f(x) - p_n(x) = \prod_{k=0}^{n}(x - x_k)\frac{f^{(n+1)}(t)}{(n+1)!}, \tag{1}$$

for some t in the smallest interval containing x_0, x_n and x. We want to find points $x_0, \ldots, x_n$ that minimize this error on a given interval $[a, b]$. Since t depends on them as well, it is in general very hard to do. The next best thing is to instead minimize only the product

$$\prod_{k=0}^{n}(x - x_k).$$

Strangely, the minimum is not achieved when the points are equidistant.

We now assume that the points $x_0, \ldots, x_n$ are chosen in the interval $[-1, 1]$. We can translate everything back to points $\tilde{x}_k$ in $[a, b]$ by making the change of variable:

$$\tilde{x}_k = \frac{(b-a)x_k + (a+b)}{2}.$$

It can be shown that we have the following lower bound on a product of monomials of this form:

$$\max_{x\in[-1,1]}\left|\prod_{k=0}^{n}(x - x_k)\right| \geq 2^{-n}.$$

We thus want to find points $x_0, \ldots, x_n$ such that the maximum on the product is exactly 2^{-n} on the interval $[-1, 1]$. The strategy is to use Chebyshev polynomials.

Definition: For $x \in [-1, 1]$, the *n-th Chebyshev polynomial* $T_n(x)$ is defined as

$$T_n(x) = \cos(n \arccos(x)).$$

We see directly that $T_0(x) = 1$ and $T_1(x) = x$. From the trigonometric identity

$$\cos(n\theta) = 2\cos(\theta)\cos((n-1)\theta) - \cos((n-2)\theta),$$

and letting $\theta = \arccos x$, we obtain the recursive formula:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

It is a polynomial of degree n such that

$$\max_{x\in[-1,1]} |T_n(x)| \leq 1,$$

since it is defined as a cosine function. We also note that the coefficient in front of x^{n+1} in $T_{n+1}(x)$ is 2^n, which can be seen from the recursive formula. Thus, factorizing $T_{n+1}(x)$ as a product of monomials, we obtain

$$T_{n+1}(x) = 2^n \prod_{k=0}^{n} (x - \alpha_k),$$

where the α_k are the zeros of T_{n+1}. Moreover,

$$\max_{x\in[-1,1]} \left| \prod_{k=0}^{n} (x - \alpha_k) \right| = \max_{x\in[-1,1]} 2^{-n} |T_{n+1}(x)| \leq 2^{-n}.$$

But since it cannot be less than 2^{-n}, we actually have

$$\max_{x\in[-1,1]} \left| \prod_{k=0}^{n} (x - \alpha_k) \right| = 2^{-n}.$$

Therefore, a minimum is achieved if we choose the points $x_0, \ldots, x_n$ to be the zeros of the Chebyshev polynomial $T_{n+1}(x)$. These are obtained by equating

$$\cos((n+1)\arccos(x)) = 0,$$

so that

$$\arccos(x_k) = \left(k + \frac{1}{2}\right)\frac{\pi}{n+1}, \qquad 0 \leq k \leq n.$$

Hence,

$$x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right), \qquad 0 \leq k \leq n.$$

They are called the *Chebyshev points*. The interpolation error in the interval $[-1, 1]$ is thus bounded by

$$|\epsilon_n(x)| \leq \frac{1}{2^n(n+1)!} \max_{x\in[-1,1]} |f^{(n+1)}(x)|, \quad \text{for all } x \in [-1, 1].$$

Note that this does not give a bound on the error at a particular point of approximation, but rather in the entire interval. It is thus better to use (1) to get a bound on the error at a particular point. For the Chebyshev points in a general interval $[a, b]$, the error is bounded by

$$|\epsilon_n(x)| \leq \frac{1}{2^n(n+1)!} \left|\frac{b-a}{2}\right|^{n+1} \max_{x\in[a,b]} |f^{(n+1)}(x)|, \quad \text{for all } x \in [a, b].$$

Example: Let $f(x) = \sin(x)$. Find the polynomial of degree 2 which interpolates $f(x)$ at the Chebyshev points in the interval $[0, 4]$. Find a bound for the error in that interval.

The roots of the polynomial $T_3(x)$ are given by

$$y_k = \cos\left(\frac{2k+1}{6}\pi\right), \qquad 0 \leq k \leq 2.$$

These are $y_0 = -\frac{\sqrt{3}}{2}$, $y_1 = 0$ and $y_2 = \frac{\sqrt{3}}{2}$. We now make a change of variables to get the points in the interval $[0, 4]$. We obtain

$$x_0 = \frac{1}{2}\left[4\left(-\frac{\sqrt{3}}{2}\right) + 4\right] = -\sqrt{3} + 2, \quad x_1 = 2, \quad x_2 = \sqrt{3} + 2.$$

Moreover,

$$f_0 = f(x_0) \approx 0.2648, \quad f_1 = f(x_1) \approx 0.9093, \quad f_2 = f(x_2) \approx -0.5568.$$

Using Newton's divided difference interpolation, we compute

$$\begin{aligned} f[x_0, x_1] &= \frac{f_1 - f_0}{x_1 - x_0} \approx 0.3721 \\ f[x_1, x_2] &= \frac{f_2 - f_1}{x_2 - x_1} \approx -0.8465 \\ f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \approx -0.3518. \end{aligned}$$

Thus,

$$\begin{aligned} f(x) &\approx 0.2648 + 0.3721(x + \sqrt{3} - 2) - 0.3518(x + \sqrt{3} - 2)(x - 2) \\ &\approx -0.0234 + 1.1610x - 0.3518x^2. \end{aligned}$$

The error term is bounded by

$$|\epsilon_2(x)| \leq \frac{1}{2^2 3!}\left|\frac{4}{2}\right|^3 \max_{x\in[0,4]} |f^{(3)}(x)| \leq \frac{1}{3} \approx 0.3333 \quad \text{for all } x \in [0, 4].$$