# Fourier Transform

| $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, \mathrm{d}\omega$ | $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, \mathrm{d}x$ |
|---|---|
| $f * g(x)$ | $\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$ |
| $f'(x)$ | $i\omega\hat{f}(\omega)$ |
| $e^{-ax^2}$ | $\frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$ |
| $e^{-a\|x\|}$ | $\sqrt{\frac{2}{\pi}} \frac{a}{\omega^2 + a^2}$ |
| $\frac{1}{x^2 + a^2}$ | $\sqrt{\frac{\pi}{2}} \frac{e^{-a\|\omega\|}}{a}$ |
| $f(x) = 1$ for $\|x\| < a$, 0 otherwise | $\sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}$ |

# Laplace Transform

| $f(t)$ | $F(s) = \int_0^{\infty} e^{-st} f(t) \, \mathrm{d}t$ |
|---|---|
| $f'(t)$ | $sF(s) - f(0)$ |
| $tf(t)$ | $-F'(s)$ |
| $e^{at}f(t)$ | $F(s-a)$ |
| $\cos(\omega t)$ | $\frac{s}{s^2 + \omega^2}$ |
| $\sin(\omega t)$ | $\frac{\omega}{s^2 + \omega^2}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $f(t-a)u(t-a)$ | $e^{-sa}F(s)$ |
| $\delta(t-a)$ | $e^{-as}$ |
| $f * g(t)$ | $F(s)G(s)$ |

# Numerics

- Newton's method: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$.
- Newton's method for system of equations: $\vec{x}_{k+1} = \vec{x}_k - JF(\vec{x}_k)^{-1}F(\vec{x}_k)$, with $JF = (\partial_j f_i)$.
- Lagrange interpolation: $p_n(x) = \sum_{k=0}^{n} \frac{l_k(x)}{l_k(x_k)} f_k$, with $l_k(x) = \prod_{j\neq k}(x - x_j)$.
- Interpolation error: $\epsilon_n(x) = \prod_{k=0}^{n}(x - x_k)\frac{f^{(n+1)}(t)}{(n+1)!}$.
- Chebyshev points: $x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right)$, $0 \leq k \leq n$.
- Newton's divided difference: $f(x) \approx f_0 + (x - x_0)f[x_0, x_1]+$ $(x-x_0)(x-x_1)f[x_0, x_1, x_2] + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f[x_0, \ldots, x_n]$, with $f[x_0, \ldots, x_k] = \frac{f[x_1,\ldots x_k]-f[x_0,\ldots,x_{k-1}]}{x_k - x_0}$.
- Trapezoid rule: $\int_a^b f(x)\ \mathrm{d}x \approx h\left[\frac{1}{2}f(a) + f_1 + f_2 + \cdots + f_{n-1} + \frac{1}{2}f(b)\right]$. Error of the trapezoid rule: $|\epsilon| \leq \frac{b-a}{12}h^2 \max_{x\in[a,b]}|f''(x)|$.
- Simpson rule: $\int_a^b f(x)\ \mathrm{d}x \approx \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n\right]$. Error of the Simpson rule: $|\epsilon| \leq \frac{b-a}{180}h^4 \max_{x\in[a,b]}|f^{(4)}(x)|$.
- Gauss-Seidel iteration: $\mathbf{x}^{(m+1)} = \mathbf{b} - \mathbf{L}\mathbf{x}^{(m+1)} - \mathbf{U}\mathbf{x}^{(m)}$, with $\mathbf{A} = \mathbf{I} + \mathbf{L} + \mathbf{U}$.
- Jacobi iteration: $\mathbf{x}^{(m+1)} = \mathbf{b} + (\mathbf{I} - \mathbf{A})\mathbf{x}^{(m)}$.
- Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$.
- Improved Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(x_n, \mathbf{y}_n) + \mathbf{f}(x_n + h, \mathbf{y}^*_{n+1})]$, where $\mathbf{y}^*_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$.
- Classical Runge-Kutta method: $\mathbf{k}_1 = h\mathbf{f}(x_n, \mathbf{y}_n)$, $\mathbf{k}_2 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_1/2)$, $\mathbf{k}_3 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_2/2)$, $\mathbf{k}_4 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_3)$, $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{6}\mathbf{k}_4$.
- Backward Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_{n+1}, \mathbf{y}_{n+1})$.
- Finite differences: $\frac{\partial u}{\partial x}(x, y) \approx \frac{u(x+h,y)-u(x-h,y)}{2h}$, $\frac{\partial^2 u}{\partial x^2}(x, y) \approx \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}$.
- Crank-Nicolson method for the heat equation: $r = \frac{k}{h^2}$, $(2 + 2r)u_{i,j+1} - r(u_{i+1,j+1} + u_{i-1,j+1}) = (2 - 2r)u_{ij} + r(u_{i+1,j} + u_{i-1,j})$.