Suggested solutions, TMA4125 Calculus 4N

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May 29th 2017

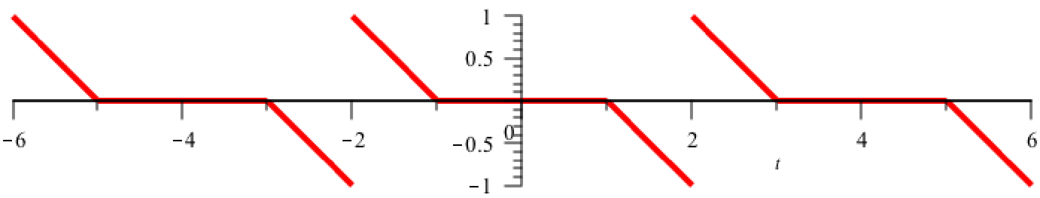
1.

The graph of $g(x)$ is displayed below. We have

$$
\begin{aligned}
b_n &= \frac{2}{2}\int_0^2 (1-x)\sin\frac{n\pi x}{2}dx \\
&= \int_1^2 \sin\frac{n\pi x}{2}dx - \int_1^2 x\sin\frac{n\pi x}{2}dx \\
&= \left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_1^2 - \left[-\frac{2}{n\pi}x\cos\frac{n\pi x}{2}\right]_1^2 + \int_1^2 -\frac{2}{n\pi}\cos\frac{n\pi x}{2}dx \\
&= \frac{2}{n\pi}\left(-\cos n\pi + \cos\frac{n\pi}{2} + 2\cos n\pi - \cos\frac{n\pi}{2}\right) + \left[-\frac{4}{(n\pi)^2}\sin\frac{n\pi x}{2}\right]_1^2 \\
&= \frac{2}{n\pi}\cos n\pi + \frac{4}{(n\pi)^2}\sin\frac{n\pi}{2}
\end{aligned}
$$

Now $\cos n\pi = (-1)^n$, and moreover $\sin\frac{n\pi}{2}$ is zero if n is even, whilst $\sin\frac{(2k+1)\pi}{2} = (-1)^k$, we may write the Fourier series of g as

$$
g(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2} + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\frac{(2k+1)\pi x}{2}
$$



2. We notice the equation may be written as

$$y' = 2\sin t + y * \cos t$$

Now $\mathcal{L}(y') = sY(s) - y(0) = sY - 1$. Using the tables of Laplace transforms for $\sin t$ and $\cos t$ and the result $\mathcal{L}(y * \cos t) = Y\mathcal{L}(\cos t)$, we find

$$sY - 1 = \frac{2}{s^2+1} + Y\frac{s}{s^2+1}$$

We rearrange to find

$$Y(\frac{s^3}{s^2+1}) = \frac{s^2+3}{s^2+1},$$

from which we deduce $Y(s) = s^{-1} + 3s^{-3}$. Using the table of inverse Laplace transforms we find

$$y(t) = 1 + \frac{3}{2}t^2$$

3. a) The idea is to use the identity $\mathcal{F}(f'(x)) = iw\hat{f}(w)$. For this purpose, we define

$$f(x) = C + \int_{-\infty}^{x} g(x)dx,$$

where C is an appropriately chosen constant. Then $f'(x) = g(x)$, and hence $\hat{f}(w) = \frac{\hat{g}(w)}{iw}$, from which we find

$$f(x) = \mathcal{F}^{-1}\left(\frac{\hat{g}(w)}{iw}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{g}(w)}{iw}e^{iwx}dw$$

In particular we have

$$\int_a^b g(x)dx = f(b) - f(a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{g}(w)(e^{iwb} - e^{iwa})}{iw}dw$$

b) Taking Fourier transforms with respect to x gives

$$\hat{u}_{tt}(w,t) = -c^2w^2\hat{u}(w,t), \quad \hat{u}(w,0) = 0,\ \hat{u}_t(x,0) = \hat{g}(w),$$

where we have also taken Fourier transforms of the initial conditions. As there are no derivatives with respect to w in the equation, we can solve like an ODE, obtaining

$$\hat{u} = A(w)\cos cwt + B(w)\sin cwt$$

Now $\hat{u}(w,0) = 0 \Rightarrow A = 0$. We then differentiate:

$$\hat{u}_t = cwB\cos cwt$$

and hence $\hat{u}_t(w,0) = cwB$. Setting $cwB(w) = \hat{g}(w)$ to satisfy the other initial condition then gives

$$\hat{u} = \frac{\hat{g}(w)}{cw}\sin cwt$$

c) We use the inverse Fourier transform to find

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{g}(w)}{cw}\sin(cwt)\, e^{iwx}dw$$

Using the identity $\sin cwt = \frac{e^{icwt} - e^{icwt}}{2i}$, we find

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{g}(w)(e^{iw(x+ct)} - e^{iw(x-ct)})}{2icw}dw$$

Applying the identity from part a) then gives

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(x')dx'$$

4. a) We write $u = F(x)G(y)$, and thus

$$u_{xx} + u_{yy} = F''G + FG'' = 0,$$

Rearranging gives

$$\frac{F''}{F} = -\frac{G''}{G},$$

and as the left hand side is a function of x alone, and the right hand side is a function of y alone, they must both be equal to some constant, k, i.e

$$F'' - kF = 0, \quad G'' + kG = 0$$

The boundary conditions give (for non-trivial solutions) $F(0) = F'(2) = 0$. We solve the equation for F subject to these conditions. There are three possible cases:

1. $k = p^2 > 0$. Then

$$F(x) = A\cosh px + B\sinh px,$$

   whereupon $F(0) = 0 \Rightarrow A = 0$. It follows that $F'(x) = Bp\cosh px$. As cosh is never zero, the condition $F'(2) = 0$ gives $B = 0$, and hence we obtain only the trivial solution.

2. $k = 0$. Then
$$F(x) = Ax + B$$
and $F(0) = 0$ gives $B = 0$. However, $F'(x) = A$, and hence $F'(2) = 0$ forces $A = 0$, giving again only the trivial solution

3. $k = -p^2 < 0$. Now
$$F(x) = A\cos px + B\sin px$$
As $F(0) = 0$ we have $A = 0$. We then differentiate to find $F'(x) = Bp\cos px$. To satisfy $F'(2) = 0$, we require
$$\cos 2p = 0 \Rightarrow p = \frac{(2n+1)\pi}{4},\ n = 1, 2, 3, \ldots$$

We now solve
$$G'' - p^2 G = 0$$
which has a general solution
$$G_n = A_n \cosh\frac{(2n+1)\pi y}{4} + B_n \sinh\frac{(2n+1)\pi y}{4}$$
We therefore have the general solution
$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cosh\frac{(2n+1)\pi y}{4} + B_n \sinh\frac{(2n+1)\pi y}{4}\right)\sin\frac{(2n+1)\pi x}{4}$$

b) Differentiation gives
$$u_y(x,0) = \sum_{n=1}^{\infty}\frac{(2n+1)\pi B_n}{4}\sin\frac{(2n+1)\pi x}{4} = 0,$$
hence $B_n = 0$ for all n. We then have
$$u(x,1) = \sum_{n=1}^{\infty} A_n \cosh\frac{(2n+1)\pi}{4}\sin\frac{(2n+1)\pi x}{4} = 50\sin\frac{5\pi x}{4}$$
The middle term is the Fourier series for the rightmost term, and comparing the two we see that $A_n = 0$ for $n \neq 2$, whilst
$$A_2 = \frac{50}{\cosh\frac{5\pi}{4}}$$

We then have

$$u(x,t) = \frac{50 \sin \frac{5\pi x}{4} \cosh \frac{5\pi y}{4}}{\cosh \frac{5\pi}{4}}$$

5. We rearrange the equations to give

$$\begin{aligned} x_1 &= \frac{1}{4}(3 - 2x_3) \\ x_2 &= \frac{1}{2}(-1 - x_3) \\ x_3 &= \frac{1}{4}(2 - 2x_1 - x_2) \end{aligned}$$

The Gauss-Seidel iteration is then

$$\begin{aligned} x_1^{n+1} &= \frac{1}{4}(3 - 2x_3^n) \\ x_2^{n+1} &= \frac{1}{2}(-1 - x_3^n) \\ x_3^{n+1} &= \frac{1}{4}(2 - 2x_1^{n+1} - x_2^{n+1}) \end{aligned}$$

Setting in $x_1^0 = x^0 2 = x_3^0 = 0$, we have

$$x_1^1 = \frac{3}{4},\ x_2^1 = -\frac{1}{2},\ x_3^1 = \frac{1}{4}(2 - 2.\frac{3}{4} - (-\frac{1}{2})) = \frac{1}{4}$$

Repeating the procedure we have

$$x_1^2 = \frac{1}{4}(3 - 2\frac{1}{4}) = \frac{5}{8},\ x_2^2 = \frac{1}{2}(-1 - \frac{1}{4}) = -\frac{5}{8},\ x_3^2 = \frac{1}{4}(2 - 2\frac{5}{8} - (-\frac{5}{8})) = \frac{11}{32}$$

6. a) We have

$$\int_0^1 \tan(x)dx \approx \frac{1/4}{3}(\tan 0 + 4\tan\frac{1}{4} + 2\tan\frac{1}{2} + 4\tan\frac{3}{4} + \tan 1) = 0.61648$$

b) We now use the formula

$$|\epsilon| \leq \frac{h^4}{180} \max |f^{(4)}(x)| = \frac{1}{15N^4},$$

where N is the number of integration subintervals (which must be even for Simpson's rule). In particular, to ensure $|\epsilon| \leq 10^{-6}$, we require

$$N^4 \geq \frac{1}{15}10^6$$

taking logarithms (we could also take the fourth root) gives

$$N = \exp(\frac{1}{4}\log(\frac{1}{15}10^6)) = 16.068$$

as N must be an even integer, we require $N = 18$ subintervals to ensure the requested accuracy.

7. The backward Euler formula gives

$$y_1 = y_0 + ht_1^2 \sin y_1$$

setting in $h = 2$,$y_0 = 1$ and $t_1 = 2$, we find

$$y_1 - 8\sin y_1 - 1 = 0,$$

which is a nonlinear equation for y_1. We use Newton's method to find a numerical solution to this equation, as specified in the problem. Now if we set $f(y_1)$ equal to the left hand side above, we find

$$f'(y_1) = 1 - 8\cos y_1$$

Write y_1^n for the nth Newton iterate; Newton's method is then

$$y_1^{n+1} = y_1^n - \frac{y_1^n - 8\sin y_1^n - 1}{1 - 8\cos y_1^n}$$

Setting in $y_1^0 = 1$, we find

$$y_1^1 = 1 + \frac{8\sin 1}{1 - 8\cos 1} = -1.02616$$

Repeating the procedure gives

$$y_1^2 = -1.02616 + \frac{-1.02616 - 8\sin(-1.02616) - 1}{1 - 8\cos(-1.02616)} = 0.505362$$

8. The Crank-Nicolson scheme discretizes first in space using the usual finite difference formulae, then in time using the trapezium rule. Letting $u_m(t) \approx u(mh,t)$, we have

$$\frac{\partial}{\partial t}u_m = \frac{1}{h^2}(u_{m-1} - 2u_m + u_{m+1}) + 8u_m = f_m(u)$$

This is a system of ODEs to which we apply the Trapezium rule:

$$u^{n+1} = u^n + \frac{k}{2}\big(f(u^n) + f(u^{n+1})\big),$$

where $u = (u_1, \ldots, u_N)$ and $f(u) = (f_1(u), \ldots, f_N(u))^T$. We then obtain the approximations $u_m^n \approx u(mh, nk)$, which follow

$$u_m^{n+1} = u_m^n + \frac{k}{2h^2}(u_{m-1}^n - 2u_m^n + u_{m+1}^n + u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}) + \frac{k}{2}(8u_m^n + 8u_m^{n+1})$$

Setting in $k = h = \frac{1}{4}$, this becomes

$$u_m^{n+1} = u_m^n + 2(u_{m-1}^n - 2u_m^n + u_{m+1}^n + u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}) + u_m^n + u_m^{n+1}$$

We then bring all the terms in u^{n+1} to the left, and all those in u^n to the right:

$$-2u_{m-1}^{n+1} + 4u_m^{n+1} - 2u_{m+1}^{n+1} = 2u_{m-1}^n - 2u_m^n + 2u_{m+1}^n$$

The boundary conditions are $u_0^n = 1, u_4^n = 0$, which can be combined with the above to give the following system

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \end{pmatrix} = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

The initial conditions are

$$u^0 = (u_1^0, u_2^0, u_3^0) = (\frac{3}{4}, \frac{1}{2}, \frac{1}{4}),$$

inserting this gives the linear system

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

Solving this system by, e.g. Gaussian elimination gives

$$u^1 = (u_1^1, u_2^1, u_3^1) = (\frac{7}{8}, 1, \frac{5}{8})$$

and for the next time step we have the linear system

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{pmatrix} = \begin{pmatrix} \frac{9}{4} \\ 1 \\ \frac{3}{4} \end{pmatrix}$$

Solving this system we obtain

$$u^2 = (\frac{19}{16}, \frac{5}{4}, \frac{13}{16})$$