Norwegian University of Science and Technology
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TMA4130 Matematikk 4N
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Fourier Transform Example

Here is an example done in class during week 38.

1 Find the function $f(x)$ s.t.

$$\int_{-\infty}^{\infty} f(x-t)e^{-2t^2}\,\mathrm{d}t = e^{-x^2} \qquad \text{for all } x \in \mathbb{R}.$$

The integral is, by definition, equal to the convolution $(f * g)(x)$, where $g(x) = e^{-2x^2}$. We thus want to isolate f in the equation

$$f(x) * e^{-2x^2} = e^{-x^2}.$$

In order to do so, we apply the Fourier transform on both side, so that the convolution becomes a multiplication by the convolution theorem:

$$\mathscr{F}(f(x) * e^{-2x^2}) = \mathscr{F}(e^{-x^2}) \implies \sqrt{2\pi}\mathscr{F}(f(x)) \cdot \mathscr{F}(e^{-2x^2}) = \mathscr{F}(e^{-x^2}).$$

Hence

$$\mathscr{F}(f) = \frac{1}{\sqrt{2\pi}} \frac{\mathscr{F}(e^{-x^2})}{\mathscr{F}(e^{-2x^2})}$$

We will get f by applying the inverse Fourier transform on both sides, but we first need to compute the expression on the right-hand side.

We compute more generally $\mathscr{F}(e^{-b^2x^2})$ for some $b \in \mathbb{R}$. Since the function $h(x) := e^{-b^2x^2} \to 0$ as $x \to \pm\infty$, we can compute the Fourier transform of the derivative. We have that

$$h'(x) = -2b^2xe^{-b^2x^2} = -2b^2xh(x).$$

Applying the Fourier transform on both side, and using linearity, we get

$$\mathscr{F}(h'(x)) = -2b^2\mathscr{F}(xh(x)).$$

By the properties of Fourier transform, the equation becomes

$$iw\mathscr{F}(h)(w) = -2b^2i\mathscr{F}(h)'(w).$$

This is a linear homogeneous ODE of degree 1 involving the function $\mathscr{F}(h)$, which depends on w. We know that a solution of such an ODE is of the form

$$\mathscr{F}(h)(w) = Ce^{-\frac{w^2}{4b^2}}.$$

The constant C is given by

$$C = \mathscr{F}(h)(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-b^2 v^2} e^{i \cdot 0 \cdot v} \, \mathrm{d}v = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-b^2 v^2} \, \mathrm{d}v = \frac{1}{\sqrt{2}b}.$$

The last integral is the Gaussian integral. There are many ways to evaluate it, the most famous being perhaps by computing the square of the integral and changing to polar coordinates. We thus have that

$$\mathscr{F}(h)(w) = \frac{1}{\sqrt{2}b} e^{-\frac{w^2}{4b^2}}.$$

Going back to the Fourier transform of f, we obtain

$$\mathscr{F}(f) = \frac{1}{\sqrt{2\pi}} \frac{\mathscr{F}(e^{-x^2})}{\mathscr{F}(e^{-2x^2})} = \frac{1}{\sqrt{2\pi}} \frac{\frac{1}{\sqrt{2}} e^{-\frac{w^2}{4}}}{\frac{1}{2} e^{-\frac{w^2}{8}}} = \frac{1}{\sqrt{\pi}} e^{\frac{-w^2}{8}}.$$

Now, applying the inverse Fourier transform on both sides, we have

$$f(x) = \frac{1}{\sqrt{\pi}} \cdot 2 \cdot \mathscr{F}^{-1} \left(\frac{1}{2} e^{-\frac{w^2}{8}} \right)(x),$$

where we have added $2 \cdot \frac{1}{2}$ to factor out

$$\mathscr{F}^{-1} \left(\frac{1}{2} e^{-\frac{w^2}{8}} \right)(x),$$

which is, as we have just discovered, equal to e^{-2x^2}. Therefore,

$$f(x) = \frac{2}{\sqrt{\pi}} e^{-2x^2}.$$