NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4125 Calculus 4N**

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Examination time (from–to): 09:00–13:00

Permitted examination support material: Kode C:
Bestemt, enkel kalkulator
Rottmann: Matematisk formelsamling

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- All numbered problems carry the same weight for grading.
- Good luck!

Language: English

Number of pages: 3

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Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☐
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Date Signature

Problem 1 Let $f : [0,2] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{for } 0 < x \leq 1 \\ 1 - x & \text{for } 1 < x \leq 2 \end{cases}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be the odd 4-periodic extension of f. Sketch its graph on the interval $[-6, 6]$ and compute the Fourier series of g.

Problem 2 Use Laplace transformations to solve the integro-differential equation

$$y' = 2 \sin t + \int_0^t y(\tau) \cos(t - \tau) d\tau,$$

with initial value $y(0) = 1$.

Problem 3

a) Let $a < b$ be real numbers, and $\hat{g}(w)$ be the Fourier transform of $g(x)$. Derive the identity

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{g}(w)(e^{iwb} - e^{iwa})}{iw} dw = \int_a^b g(x) dx$$

b) Take Fourier transforms with respect to x on both sides of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Suppose we wish to solve the equation with respect to the initial conditions $u(x, 0) = 0$, $u_t(x, 0) = g(x)$. By solving the equation for $\hat{u}$, show that the Fourier transform of u is given by

$$\hat{u}(w, t) = \frac{\hat{g}(w)}{cw} \sin cwt$$

c) Deduce from the above that

$$u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(x) dx$$

Problem 4

a) The steady state temperature of a plate of dimension 2×1 (in metres) is modeled by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

for $0 \leq x \leq 2$, $0 \leq y \leq 1$. The right hand side of the plate is insulated, whilst the left hand side is held at a constant temperature of 0°C, i.e.

$$u(0, y) = u_x(2, y) = 0,$$

Find all possible solutions $u(x, y) = F(x)G(y)$ of the equation satisfying the above conditions.

b) If the bottom edge of the plate is also insulated, i.e. $u_y(x, 0) = 0$, and the top edge is held at temperature

$$u(x, 1) = (50 \sin \frac{5\pi x}{4})°\text{C},$$

solve the equation to find the steady state temperature $u(x, y)$ of the plate.

Problem 5 Consider the system of linear equations

$$\begin{aligned} 4x_1 + 2x_3 &= 3 \\ 2x_2 + x_3 &= -1 \\ 2x_1 + x_2 + 4x_3 &= 2 \end{aligned}$$

Find an approximation to the solution of the above system by computing two steps of the Gauss-Seidel iteration. Use $x_1 = x_2 = x_3 = 0$ as the initial value.

Problem 6 Let $f(x) = \tan(x)$.

a) Find a numerical approximation to

$$\int_0^1 f(x)dx$$

by using Simpson's rule with $2n = 4$ integration intervals.

b) It may be shown that $|f^{(4)}(x)| \leq 12$ on $x \in [0, 1]$. How many integration intervals do we require to ensure that the error of the Simpson's rule approximation is less than 10^{-6}?

Problem 7 Consider the differential equation

$$y'(t) = t^2 \sin\Big(y(t)\Big), \quad y(0) = 1$$

Find an approximation to $y(2)$ by applying the backward Euler method with $h = 2$ and solving the resulting nonlinear equation using two iterations of Newton's method (use the value of $y(0)$ as the initial value for the Newton iterations).

Problem 8 Consider the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 8u, \quad 0 < x < 1, t > 0$$

with boundary conditions $u(0,t) = 1$, $u(1,t) = 0$ and initial condition $u(x,0) = 1 - x$.

Formulate the Crank-Nicolson method for the numerical solution of the equation. Use step sizes $h = 0.25$ in the x-direction and $k = 0.25$ in the t-direction, and perform two time-steps.

Fourier

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega$	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$
$f * g(x)$	$\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$
$f'(x)$	$i\omega\hat{f}(\omega)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$
$e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}} \frac{a}{\omega^2+a^2}$
$\frac{1}{1+x^2}$	$\sqrt{\frac{\pi}{2}} e^{-\|\omega\|}$
$f(x) = 1$ for $\|x\| < a$, 0 otherwise	$\sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}$

Laplace transform

$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\, dt$
$f'(t)$	$sF(s) - f(0)$
$tf(t)$	$-F'(s)$
$e^{at}f(t)$	$F(s-a)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$f(t-a)u(t-a)$	$e^{-sa}F(s)$
$\delta(t-a)$	e^{-as}
$f * g(t)$	$F(s)G(s)$

Numerics

- Newton's method: $x_{k+1} = x_k - f(x_k)/f'(x_k)$.

- Newton's method for systems: $\mathbf{J}^{(k)}\Big(\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\Big) = -\mathbf{f}(\mathbf{x}^{(k)})$ with $(\mathbf{J}^{(k)})_{ij} = \partial_j f_i^{(k)}$
- Lagrange interpolation polynomial: $L_k(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$, $p_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k)$
- Trapezoid rule: $\int_a^b f(x)\,dx \approx h\Big[\frac{1}{2}f(a) + f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + \frac{1}{2}f(b)\Big]$

  Error of the trapezoid rule: $|\epsilon| \le h^2 \frac{b-a}{12} \max_{a\le x\le b} |f''(x)|$.
- Simpson rule: $\int_a^b f(x)\,dx \approx \frac{h}{3}\Big[f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 2f_{n-2} + 4f_{n-1} + f_n\Big]$ with $f_i = f(x_i)$.

  Error of the Simpson rule: $|\epsilon| \le h^4 \frac{b-a}{180} \max_{a\le x\le b} |f^{(4)}(x)|$.
- Gauss–Seidel iteration: $\mathbf{x}^{(k+1)} = \mathbf{b} - \mathbf{L}\mathbf{x}^{(k+1)} - \mathbf{U}\mathbf{x}^{(k)}$ with $\mathbf{A} = \mathbf{I} + \mathbf{L} + \mathbf{U}$.
- Jacobi iteration: $\mathbf{x}^{(k+1)} = \mathbf{b} + (\mathbf{I} - \mathbf{A})\mathbf{x}^{(k)}$
- Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(t_n, \mathbf{y}_n)$
- Improved Euler method: $\mathbf{k}_1 = h\mathbf{f}(t_n, \mathbf{y}_n)$, $\mathbf{k}_2 = h\mathbf{f}(t_n + h, \mathbf{y}_n + \mathbf{k}_1)$,

  $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}\mathbf{k}_1 + \frac{1}{2}\mathbf{k}_2$.
- Classical Runge–Kutte method:

  $\mathbf{k}_1 = h\mathbf{f}(t_n, \mathbf{y}_n)$, $\mathbf{k}_2 = h\mathbf{f}(t_n + h/2, \mathbf{y}_n + \mathbf{k}_1/2)$,

  $\mathbf{k}_3 = h\mathbf{f}(t_n + h/2, \mathbf{y}_n + \mathbf{k}_2/2)$, $\mathbf{k}_4 = h\mathbf{f}(t_n + h, \mathbf{y}_n + \mathbf{k}_3)$,
- Backward Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})$
- Finite differences: $\frac{\partial u}{\partial x}(x,y) \approx \frac{u(x+h,y)-u(x-h,y)}{2h}$

  $\frac{\partial^2 u}{\partial x^2}(x,y) \approx \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}$

  $\frac{\partial u}{\partial y}(x,y) \approx \frac{u(x,y+h)-u(x,y-h)}{2h}$

  $\frac{\partial^2 u}{\partial y^2}(x,y) \approx \frac{u(x,y+h)-2u(x,y)+u(x,y-h)}{h^2}$