Problem 1 (TMA4125/TMA4130)

Use the Laplace transformation for solving the differential equation

$$y'' + 3y' + 2y = tu(t-1)$$

with the initial conditions

$$y(0) = 1, \qquad y'(0) = -1.$$

Possible Solution

Application of the Laplace-transform to the equation gives us

$$s^2Y - sy(0) - y'(0) + 3sY - 3y(0) + 2Y = \mathcal{L}(tu(t-1)).$$

The right hand side computes to

$$\begin{aligned}\mathcal{L}(tu(t-1)) = \mathcal{L}\Big(((t-1)+1)u(t-1)\Big) &= e^{-s}\mathcal{L}(t+1)\\ &= e^{-s}\Big(\frac{1}{s^2}+\frac{1}{s}\Big) = e^{-s}\frac{s+1}{s^2}.\end{aligned}$$

Inserting this and the initial conditions into the transformed equation, we obtain

$$s^2Y - s + 1 + 3sY - 3 + Y = e^{-s}\frac{s+1}{s^2}.$$

Noting that

$$(s^2 + 3s + 1) = (s+2)(s+1)$$

we obtain that

$$(s+2)(s+1)Y = s + 2 + e^{-s}\frac{s+1}{s^2}$$

and thus

$$Y = \frac{1}{s+1} + e^{-s}\frac{1}{s^2(s+2)}.$$

Now

$$\frac{1}{s^2(s+2)} = \frac{1}{2}\frac{s+2-s}{s^2(s+2)} = \frac{1}{2}\frac{1}{s^2} - \frac{1}{2}\frac{1}{s(s+2)} = \frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{s+2-s}{s(s+2)} = \frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s} + \frac{1}{4}\frac{1}{s+2}.$$

Thus

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\Big(\frac{1}{s+1} + e^{-s}\Big(\frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s} + \frac{1}{4}\frac{1}{s+2}\Big)\Big)\\ &= e^{-t} + \frac{1}{4}u(t-1)\Big(2(t-1) - 1 + e^{-(t-2)}\Big)\\ &= e^{-t} + \frac{1}{4}u(t-1)\Big(2t - 3 + e^{2-t}\Big).\end{aligned}$$

Problem 1 (TMA4122)

Let f be the function

$$f(x) = \frac{1}{1+x^2} \qquad \text{for } x \in \mathbb{R}.$$

Use the Fourier transformation for computing the convolution $(f * f)(x)$.

Possible Solution

We have

$$\mathcal{F}(f * f)(\omega) = \sqrt{2\pi}(\mathcal{F}(f)(\omega))^2 = \sqrt{2\pi}\left(\sqrt{\frac{\pi}{2}}e^{-|\omega|}\right)^2 = \sqrt{\frac{\pi^3}{2}}e^{-2|\omega|}.$$

Now we note that

$$\mathcal{F}^{-1}(e^{-2|\omega|})(x) = \sqrt{\frac{2}{\pi}}\frac{2}{4+x^2},$$

and therefore

$$f * f(x) = \sqrt{\frac{\pi^3}{2}}\mathcal{F}^{-1}(e^{-2|\omega|})(x) = \frac{2\pi}{4+x^2}.$$

Problem 1 (TMA4123)

Consider the Matlab script

```
function x = TMA4123(N)
x = 0;
for i=1:N
x = x - (exp(x)-x^2)/(exp(x)-2*x);
end
```

Compute the return value of the script for $N = 2$ and explain why this is an approximation to the solution of the equation

$$e^x = x^2.$$

Possible Solution

Newton's method for the solution of the equation $e^x - x^2 = 0$ reads

$$x_{n+1} = x_n - \frac{e^{x_n} - {x_n}^2}{e^{x_n} - 2x_n},$$

which is exactly the formula that defines the updates in the for–loop. Thus the Matlab script is nothing else than an implementation of Newton's method for the solution of the given equation with starting value $x = 0$ and N iterations.

With $N = 2$ we obtain in the first step a value of

$$x = 0 - \frac{e^0 - 0^2}{e^0 - 2 \cdot 0} = -1,$$

and in the second step

$$x = -1 - \frac{e^{-1} - (-1)^2}{e^{-1} - 2 \cdot (-1)} \approx -0.7330436.$$

This is also the return value of the script.

Problem 2

Find the polynomial of lowest degree that interpolates the points

x_i	-2	-1	0	1	2
$f(x_i)$	2	4	0	-4	4

Possible Solution

Using Newton interpolation[1], we obtain

$$\begin{array}{r||cccccccc}
-2 & 2 & & & & \\
 & & 2 & & & \\
-1 & 4 & & -3 & & \\
 & & -4 & & 1 & \\
0 & 0 & & 0 & & 1/4 \\
 & & -4 & & 2 & \\
1 & -4 & & 6 & & \\
 & & 8 & & & \\
2 & 4 & & & &
\end{array}$$

Thus the interpolation polynomial in Newton form reads

$$\begin{aligned}
p(x) &= 2 + 2(x+2) - 3(x+2)(x+1) + (x+2)(x+1)x + \frac{1}{4}(x+2)(x+1)x(x-1) \\
&= 2 + 2(x+2) - 3(x^2+3x+2) + x^3 + 3x^2 + 2x + \frac{1}{4}(x^4 + 2x^3 - x^2 - 2x) \\
&= \frac{1}{4}x^4 + \frac{3}{2}x^3 - \frac{1}{4}x^2 - \frac{11}{2}x.
\end{aligned}$$

[1] Lagrange interpolation would be fine as well

Problem 3

Use the trapezoidal rule with step length $h = 0.25$ in order to find an approximation T of the integral

$$I = \int_0^1 e^{x^2}\, dx.$$

Find an upper bound for the error $|I - T|$.

Possible Solution

The trapezoidal rule with $h = 0.25$ for this integral reads

$$T = \frac{1}{4}\left(\frac{1}{2}e^0 + e^{0.25^2} + e^{0.5^2} + e^{0.75^2} + \frac{1}{2}e^1\right) \approx 1.490679.$$

For the error estimate we use the formula

$$|\epsilon| \le \frac{1}{12} 0.25^2 \max_{0\le x\le 1} |(e^{x^2})''|.$$

Now

$$(e^{x^2})' = 2xe^{x^2}$$

and thus

$$(e^{x^2})'' = (2xe^{x^2})' = (4x^2 + 2)e^{x^2}.$$

Since both the functions $4x^2 + 2$ and e^{x^2} are positive and increasing on the interval $[0, 1]$, the maximum is attained for the largest value of x, that is, for $x = 1$. Thus

$$\max_{0\le x\le 1} |(e^{x^2})''| = 6e$$

and we obtain the error estimate

$$|\epsilon| \le \frac{1}{12} 0.25^2 6e = \frac{e}{32} \approx 0.08494631.$$

Problem 4

Perform two iterations of the Jacobi method for solving the linear system

$$\begin{aligned} 5x_1 + 2x_2 + x_3 &= 5, \\ -x_1 - 5x_2 + x_3 &= 5, \\ x_1 + x_2 + 3x_3 &= -3. \end{aligned}$$

Use the initial value $x^{(0)} = (0, 0, 0)$.

Possible Solution

The Jacobi method for this system reads

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{5}\Big(5 - 2x_2^k - x_3^k\Big), \\ x_2^{(k+1)} &= -\frac{1}{5}\Big(5 + x_1^k - x_3^k\Big), \\ x_3^{(k+1)} &= \frac{1}{3}\Big(-3 - x_1^k - x_2^k\Big). \end{aligned}$$

In the first iteration we obtain

$$\begin{aligned} x_1^{(1)} &= \frac{1}{5}\Big(5 - 2 \cdot 0 - 0\Big) = 1, \\ x_2^{(1)} &= -\frac{1}{5}\Big(5 + 0 - 0\Big) = -1, \\ x_3^{(1)} &= \frac{1}{3}\Big(-3 - 0 - 0\Big) = -1. \end{aligned}$$

In the second iteration, we obtain

$$\begin{aligned} x_1^{(2)} &= \frac{1}{5}\Big(5 - 2 \cdot (-1) - (-1)\Big) = \frac{8}{5}, \\ x_2^{(2)} &= -\frac{1}{5}\Big(5 + 1 - (-1)\Big) = -\frac{7}{5}, \\ x_3^{(2)} &= \frac{1}{3}\Big(-3 - 1 - (-1)\Big) = -1. \end{aligned}$$

Problem 5

Let f be the 6-periodic function defined by

$$f(x) = x + 3 \qquad \text{for } -3 < x < 3.$$

Find the Fourier series of f.

Possible Solution

The Fourier coefficient a_0 computes as

$$a_0 = \frac{1}{6}\int_{-3}^{3} x + 3 \, dx = 3.$$

If $n \geq 1$, we obtain

$$a_n = \frac{1}{3}\int_{-3}^{3} (x+3)\cos\frac{n\pi x}{3} \, dx.$$

Now note that the function x is odd, while cos is even, which implies that $x \cos \frac{n\pi x}{3}$ is odd, and thus its integral from -3 to 3 equal to 0. Thus

$$a_n = \frac{1}{3} \int_{-3}^{3} 3 \cos \frac{n\pi x}{3} \, dx = \frac{3}{n\pi} \sin \frac{n\pi x}{3} \Big|_{-3}^{3} = 0.$$

For the computation of the coefficients b_n we obtain

$$b_n = \frac{1}{3} \int_{-3}^{3} (x+3) \sin \frac{n\pi x}{3} \, dx,$$

which reduces, because sin is odd, to

$$b_n = \frac{1}{3} \int_{-3}^{3} x \sin \frac{n\pi x}{3} \, dx,$$

Integration by parts yields

$$\begin{aligned} b_n &= -\frac{1}{3} x \frac{3}{n\pi} \cos \frac{n\pi x}{3} \Big|_{-3}^{3} - \frac{1}{3} \int_{-3}^{3} \left(-\frac{3}{n\pi} \cos \frac{n\pi x}{3} \right) dx \\ &= -\frac{1}{n\pi} \Big(3(-1)^n - (-3)(-1)^n \Big) + \frac{1}{n\pi} \frac{3}{n\pi} \sin \frac{n\pi x}{3} \Big|_{-3}^{3} \\ &= \frac{6(-1)^{n+1}}{n\pi} \end{aligned}$$

Thus we obtain the Fourier series expansion

$$f(x) = 3 + \sum_{n=1}^{\infty} \frac{6(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{3}.$$

Problem 6

Let f be the 2π-periodic function given by

$$f(x) = \begin{cases} e^x & \text{for } 0 < x < \pi, \\ 0 & \text{for } -\pi < x < 0. \end{cases}$$

Assume that a_n and b_n are the Fourier coefficients of f, and denote by g and h the functions

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \qquad \text{and} \qquad h(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

Sketch the graphs of the functions f, g, and h on the interval $[-2\pi, 2\pi]$, and find the values of $f(x)$ and $g(x)$ in the points $x = -\pi/2$, $x = 0$, and $x = \pi/2$.

Possible Solution

Since the function g contains only a constant and cosine terms, it is even, that is $g(x) = g(-x)$. Also, the function h only contains sine terms and therefore is odd, that is, $h(-x) = -h(x)$. Finally, we note that

$$g(x) + h(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big) = f(x)$$

(at all points x where f is continuous), as a_n and b_n are the Fourier coefficients of f. In other words,

$$\begin{aligned} g(x) + h(x) &= e^x && \text{for all } 0 < x < \pi, \\ g(x) + h(x) &= 0 && \text{for all } -\pi < x < 0. \end{aligned}$$

Now the second equation can also be written as

$$g(-x) + h(-x) = 0 \qquad \text{for all } 0 < x < \pi.$$

Using the fact that $g(-x) = g(x)$ and $h(-x) = -h(x)$, we thus obtain the system of equations

$$\begin{aligned} g(x) + h(x) &= e^x && \text{for all } 0 < x < \pi, \\ g(x) - h(x) &= 0 && \text{for all } 0 < x < \pi. \end{aligned}$$

Solving this system for $g(x)$ and $h(x)$ yields

$$g(x) = h(x) = \frac{1}{2} e^x \qquad \text{for all } 0 < x < \pi.$$

Since g is even and h is odd, this means that

$$g(x) = \begin{cases} \frac{1}{2} e^x & \text{for } 0 < x < \pi, \\ \frac{1}{2} e^{-x} & \text{for } -\pi < x < 0, \end{cases}$$

and

$$h(x) = \begin{cases} \frac{1}{2} e^x & \text{for } 0 < x < \pi, \\ -\frac{1}{2} e^{-x} & \text{for } -\pi < x < 0. \end{cases}$$

In particular, we obtain

$$g(-\pi/2) = g(\pi/2) = \frac{1}{2} e^{\pi/2}$$

and

$$h(-\pi/2) = -\frac{1}{2} e^{\pi/2} \qquad \text{and} \qquad h(\pi/2) = \frac{1}{2} e^{\pi/2}.$$

At the point 0, the functions g and h are equal to the means of their left-hand and right-hand limits, respectively. Therefore

$$g(0) = \frac{1}{2}\Big(g(0-0) + g(0+0)\Big) = \frac{1}{2}\Big(e^{-0} + e^{0}\Big) = 1$$

and

$$h(x) = \frac{1}{2}\Big(h(0-0) + h(0+0)\Big) = \frac{1}{2}\Big(-e^{-0} + e^{0}\Big) = 0.$$

Problem 7

We want to find a numerical solution of the partial differential equation

$$\frac{\partial u}{\partial t}(x,t) = tu(x,t) + \frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 \le x \le 1, \qquad t > 0,$$

with boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 1 \qquad \text{for all } t > 0$$

and initial condition

$$u(x,0) = x \qquad \text{for } 0 \le x \le 1.$$

Formulate an explicit method for solving this partial differential equation with the given boundary and initial conditions.

Use a step length of $h = 1/4$ in space and perform two time steps of length $k = 1/10$.

Possible Solution

Denote $x_i := ih$ and $t_j := jk$, and let $u_{i,j}$ be a numerical approximation of $u(x_i, t_j)$.

The left hand side of the PDE can be discretised (in order to obtain an explicit method) as

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{u_{i,j+1} - u_{i,j}}{k},$$

and the right hand side as

$$t_j u(x_i, t_j) + \frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx jku_{i,j} + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}.$$

Thus one obtains

$$\frac{u_{i,j+1} - u_{i,j}}{k} = jku_{i,j} + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

or, explicitly,

$$u_{i,j+1} = u_{i,j} + k\Big(jku_{i,j} + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\Big).$$

With $h = 1/4$ and $k = 1/10$, this becomes

$$u_{i,j+1} = u_{i,j} + \frac{j}{100}u_{i,j} + \frac{8}{5}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}).$$

Now we use the initial condition $u(x,0) = x$ in order to obtain that

$$\begin{aligned} u_{0,0} &= 0, \\ u_{1,0} &= \frac{1}{4}, \\ u_{2,0} &= \frac{1}{2}, \\ u_{3,0} &= \frac{3}{4}, \\ u_{4,0} &= 1. \end{aligned}$$

Thus

$$\begin{aligned} u_{1,1} &= u_{1,0} + 0 + \frac{8}{5}(u_{0,0} - 2u_{1,0} + u_{2,0}) = \frac{1}{4}, \\ u_{2,1} &= u_{2,0} + 0 + \frac{8}{5}(u_{1,0} - 2u_{2,0} + u_{3,0}) = \frac{1}{2}, \\ u_{3,1} &= u_{3,0} + 0 + \frac{8}{5}(u_{2,0} - 2u_{3,0} + u_{4,0}) = \frac{3}{4}. \end{aligned}$$

For the next iteration, we use the boundary conditions

$$u_{0,1} = 0 \qquad \text{and} \qquad u_{4,1} = 1,$$

and obtain

$$\begin{aligned} u_{1,2} &= u_{1,1} + \frac{1}{100}u_{1,1} + \frac{8}{5}(u_{0,1} - 2u_{1,1} + u_{2,1}) = \frac{101}{400} = 0.2525, \\ u_{2,2} &= u_{2,1} + \frac{1}{100}u_{2,1} + \frac{8}{5}(u_{1,1} - 2u_{2,1} + u_{3,1}) = \frac{101}{200} = 0.505, \\ u_{3,2} &= u_{3,1} + \frac{1}{100}u_{3,1} + \frac{8}{5}(u_{2,1} - 2u_{3,1} + u_{4,1}) = \frac{303}{400} = 0.7575. \end{aligned}$$

Problem 8

Use the Fourier transformation for solving the partial differential equation

$$\frac{\partial u}{\partial t}(x,t) = t\frac{\partial^2 u}{\partial x^2}(x,t), \qquad x \in \mathbb{R}, \qquad t > 0,$$

with initial condition

$$u(x,0) = e^{-\frac{x^2}{2}}, \qquad x \in \mathbb{R}.$$

Possible Solution

We apply the Fourier transformation (with respect only to the x-variable) to the given PDE and obtain

$$\frac{\partial \hat{u}}{\partial t}(\omega, t) = -t\omega^2 \hat{u}(\omega, t).$$

For every $\omega \in \mathbb{R}$, this is an ODE with respect to t, with solution

$$\hat{u}(\omega, t) = C(\omega) e^{-\frac{\omega^2 t^2}{2}}.$$

In order to find $C(\omega)$, we use the initial condition, which, after a Fourier transformation, reads as

$$\hat{u}(\omega, 0) = e^{-\frac{\omega^2}{2}}.$$

Thus

$$C(\omega) = e^{-\frac{\omega^2}{2}},$$

and we obtain

$$\hat{u}(\omega, t) = e^{-\frac{\omega^2}{2}} e^{-\frac{\omega^2 t^2}{2}} = e^{-\omega^2 \frac{t^2+1}{2}}.$$

Finally we compute the inverse Fourier transform of $\hat{u}$ and obtain

$$u(x, t) = \mathcal{F}^{-1}(\hat{u}(\omega, t)) = \frac{1}{\sqrt{t^2+1}} e^{-\frac{x^2}{2(t^2+1)}}.$$

Problem 9a

Given the equation

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) + 5u(x, y) = 0, \qquad 0 < x < \pi, \quad 0 < y < \pi/4,$$

find all solutions of the form $u(x, y) = F(x)G(y)$ that satisfy the boundary conditions

$$u(0, y) = 0 \qquad \text{and} \qquad u(\pi, y) = 0, \qquad\qquad 0 < y < \pi/4.$$

Possible Solution

Inserting the function $u(x, y) = F(x)G(y)$ into the PDE, we obtain the equation

$$F''(x)G(y) + F(x)G''(y) + 5F(x)G(y) = 0,$$

which we rewrite as
$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} - 5.$$
Now the left hand side only depends on x while the right hand side only depends on y, which implies that both sides need to be constant, say equal to k for some $k \in \mathbb{R}$. We thus obtain that
$$\begin{aligned} F''(x) &= kF(x), \\ G''(y) &= (5-k)G(y). \end{aligned}$$

We now consider the different possibilities for k.

- $k = 0$:

  Here the equation for F has the solution
  $$F(x) = Ax + B.$$
  The boundary condition $F(0) = 0$ implies that $B = 0$, and the boundary condition $F(\pi) = 0$ then implies that $0 = A\pi$, and thus $A = 0$. Thus we only obtain the trivial solution $F(x) = 0$.

- $k = p^2 > 0$:

  Here the equation for F has the solution
  $$F(x) = A \sinh px + B \cosh px.$$
  Now the boundary condition $F(0) = 0$ implies that $B = 0$, and the boundary condition $F(\pi) = 0$ then implies that $0 = A \sinh p\pi$, and therefore $A = 0$. Again, we only obtain the trivial solution.

- $k = -p^2 < 0$:

  Here we obtain for F the solutions
  $$F(x) = A \sin px + B \cos px.$$
  Again the boundary condition $F(0) = 0$ implies that $B = 0$. The boundary condition $F(\pi) = 0$, however, then implies that either $A = 0$ or $p \in \mathbb{N}$. Thus we obtain the non-trivial solution
  $$F(x) = \sin px$$
  with $p \in \mathbb{N}$.

Now we turn to the solution of the equation

$$G''(y) = (5-k)G(y) = (5+p^2)G(y),$$

which has the solution

$$G(y) = A\cosh(\sqrt{5+p^2}y) + B\sinh(\sqrt{5+p^2}y).$$

In total, we therefore obtain the nontrivial solutions

$$u(x,y) = \sin(px)\Big(A\cosh(\sqrt{5+p^2}y) + B\sinh(\sqrt{5+p^2}y)\Big)$$

for $p \in \mathbb{N}$.

Problem 9b

Find the solution of the problem in part **a)** that in addition satisfies the boundary conditions

$$\begin{aligned} u(x,0) &= \sin(x), \qquad 0 < x < \pi, \\ u(x,\pi/4) &= \sin(x), \qquad 0 < x < \pi. \end{aligned}$$

Possible Solution

Using the results of Problem 9a, we obtain that

$$u(x,y) = \sum_{p=1}^{\infty} \sin(px)\Big(A_p\cosh(\sqrt{5+p^2}y) + B_p\sinh(\sqrt{5+p^2}y)\Big)$$

for some coefficients $A_p,\ B_p \in \mathbb{R}$, $p \in \mathbb{N}$.

The boundary condition $u(x,0) = \sin(x)$ now yields that

$$\sin x = u(x,0) = \sum_{p=1}^{\infty} \sin(px)A_p\,,$$

which immediately implies that

$$A_1 = 1 \qquad \text{and } A_p = 0 \text{ for } p \geq 2.$$

That is,

$$u(x,y) = \sin(x)\cosh(\sqrt{6}y) + \sum_{p=1}^{\infty} B_p\sin(px)\sinh(\sqrt{5+p^2}y).$$

Now we insert the second boundary condition $u(x, \pi/2) = \sin(x)$ and obtain that

$$\sin(x) = \sin(x)\cosh(\sqrt{6}\pi/2) + B_1 \sin(x)\sinh(\sqrt{6}\pi/2) + \sum_{p=2}^{\infty} B_p \sin(px)\sinh(\sqrt{5+p^2}y).$$

This implies that $B_p = 0$ for $p \geq 2$ and

$$B_1 = -\frac{\cosh(\sqrt{6}\pi/2)}{\sinh(\sqrt{6}\pi/2)}$$

Thus

$$u(x,y) = \sin x \cosh(\sqrt{6}y) - \frac{\cosh(\sqrt{6}\pi/2)}{\sinh(\sqrt{6}\pi/2)} \sin x \sinh(\sqrt{6}y).$$