NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
TMA4122/TMA4123/TMA4125/TMA4130 Matematikk 4M/N

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Examination time (from–to):

Permitted examination support material: Code C): Basic calculator. Rottmann: *Mathematical formulæ*

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- All sub-problems carry the same weight for grading.
- Good luck!

Language: English

Number of pages: 3

Number pages enclosed: 2

Checked by:

Date Signature

Problem 1 Only for TMA4125/TMA4130 Matematikk 4N!

Use the Laplace transformation for solving the differential equation

$$y'' + 3y' + 2y = tu(t-1)$$

with the initial conditions

$$y(0) = 1, \qquad y'(0) = -1.$$

Problem 1 Only for TMA4122 Matematikk 4M!

Let f be the function

$$f(x) = \frac{1}{1+x^2} \qquad \text{for } x \in \mathbb{R}.$$

Use the Fourier transformation for computing the convolution $(f * f)(x)$.

Problem 1 Only for TMA4123 Matematikk 4M!

Consider the Matlab script

```
function x = TMA4123(N)
x = 0;
for i=1:N
x = x - (exp(x)-x^2)/(exp(x)-2*x);
end
```

Compute the return value of the script for $N = 2$ and explain why this is an approximation to the solution of the equation

$$e^x = x^2.$$

Problem 2 Find the polynomial of lowest degree that interpolates the points

x_i	-2	-1	0	1	2
$f(x_i)$	2	4	0	-4	4

Problem 3 Use the trapezoidal rule with step length $h = 0.25$ in order to find an approximation T of the integral

$$I = \int_0^1 e^{x^2}\, dx.$$

Find an upper bound for the error $|I - T|$.

Problem 4 Perform two iterations of the Jacobi method for solving the linear system

$$\begin{aligned} 5x_1 + 2x_2 + x_3 &= 5, \\ -x_1 - 5x_2 + x_3 &= 5, \\ x_1 + x_2 + 3x_3 &= -3. \end{aligned}$$

Use the initial value $x^{(0)} = (0, 0, 0)$.

Problem 5 Let f be the 6-periodic function defined by

$$f(x) = x + 3 \qquad \text{for } -3 < x < 3.$$

Find the Fourier series of f.

Problem 6 Let f be the 2π-periodic function given by

$$f(x) = \begin{cases} e^x & \text{for } 0 < x < \pi, \\ 0 & \text{for } -\pi < x < 0. \end{cases}$$

Assume that a_n and b_n are the Fourier coefficients of f, and denote by g and h the functions

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \qquad \textit{og} \qquad h(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

Sketch the graphs of the functions f, g, and h on the interval $[-2\pi, 2\pi]$, and find the values of $f(x)$ and $g(x)$ in the points $x = -\pi/2$, $x = 0$, and $x = \pi/2$.

Problem 7 We want to find a numerical solution of the partial differential equation

$$\frac{\partial u}{\partial t}(x,t) = tu(x,t) + \frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 \le x \le 1, \qquad t > 0,$$

with boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 1 \qquad \text{for all } t > 0$$

and initial condition

$$u(x,0) = x \qquad \text{for } 0 \le x \le 1.$$

Formulate an explicit method for solving this partial differential equation with the given boundary and initial conditions.

Use a step length of $h = 1/4$ in space and perform two time steps of length $k = 1/10$.

Problem 8 Use the Fourier transformation for solving the partial differential equation

$$\frac{\partial u}{\partial t}(x,t) = t\frac{\partial^2 u}{\partial x^2}(x,t), \qquad x \in \mathbb{R}, \qquad t > 0,$$

with initial condition

$$u(x,0) = e^{-\frac{x^2}{2}}, \qquad x \in \mathbb{R}.$$

Problem 9

a) Given the equation

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) + 5u(x,y) = 0, \qquad 0 < x < \pi, \quad 0 < y < \pi/4,$$

find all solutions of the form $u(x,y) = F(x)G(y)$ that satisfy the boundary conditions

$$u(0,y) = 0 \qquad \text{and} \qquad u(\pi,y) = 0, \qquad 0 < y < \pi/4.$$

b) Find the solution of the problem in part **a)** that in addition satisfies the boundary conditions

$$\begin{aligned} u(x,0) &= \sin(x), \qquad 0 < x < \pi, \\ u(x,\pi/4) &= \sin(x), \qquad 0 < x < \pi. \end{aligned}$$

Fourier

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega$	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$
$f * g(x)$	$\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$
$f'(x)$	$i\omega \hat{f}(\omega)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$
$e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}} \frac{a}{\omega^2+a^2}$
$\frac{a}{a^2+x^2}$	$\sqrt{\frac{\pi}{2}} e^{-a\|\omega\|}$
$f(x) = 1$ for $\|x\| < a$, 0 otherwise	$\sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}$

Laplace transform

$f(t)$	$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$
$e^{at} f(t)$	$F(s-a)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$f(t-a)u(t-a)$	$e^{-sa}F(s)$
$\delta(t-a)$	e^{-as}

Numerics

- Newton's method: $x_{k+1} = x_k - f(x_k)/f'(x_k)$
- Newton's method for systems: $\mathbf{JF}(\mathbf{x}^{(k)})\mathbf{h}^{(k)} = -\mathbf{F}(\mathbf{x}^{(k)})$ and $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{h}^{(k)}$ with $(\mathbf{JF}(\mathbf{x}^{(k)}))_{ij} = \partial_j f_i(\mathbf{x}^{(k)})$
- Lagrange interpolation polynomial: $L_k(x) = \frac{(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$,
  $p_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k)$
- Trapezoid rule: $\int_a^b f(x)\,dx \approx h\left[\frac{1}{2}f_0 + f_1 + f_2 + \ldots + f_{n-1} + \frac{1}{2}f_n\right]$

  Error of the trapezoid rule: $|\epsilon| \leq \frac{b-a}{12}h^2 \max_{a\leq x\leq b}|f''(x)|$.
- Simpson rule: $\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 2f_{n-2} + 4f_{n-1} + f_n\right]$ with $f_i = f(x_i)$.

  Error of the Simpson rule: $|\epsilon| \leq \frac{b-a}{180}h^4 \max_{a\leq x\leq b}|f^{(4)}(x)|$.
- Gauß–Seidel iteration: $\mathbf{x}^{(k+1)} = \mathbf{b} - \mathbf{L}\mathbf{x}^{(k+1)} - \mathbf{U}\mathbf{x}^{(k)}$ with $\mathbf{A} = \mathbf{I} + \mathbf{L} + \mathbf{U}$.
- Jacobi iteration: $\mathbf{x}^{(k+1)} = \mathbf{b} + (\mathbf{I} - \mathbf{A})\mathbf{x}^{(k)}$
- Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$
- Improved Euler method: $\mathbf{k}_1 = h\mathbf{f}(x_n, \mathbf{y}_n)$, $\mathbf{k}_2 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_1)$,
  $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}\mathbf{k}_1 + \frac{1}{2}\mathbf{k}_2$.
- Classical Runge–Kutte method:
  $\mathbf{k}_1 = h\mathbf{f}(x_n, \mathbf{y}_n)$, $\mathbf{k}_2 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_1/2)$,
  $\mathbf{k}_3 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_2/2)$, $\mathbf{k}_4 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_3)$,
  $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{6}\mathbf{k}_4$.
- Backward Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_{n+1}, \mathbf{y}_{n+1})$
- Finite differences:
  $\frac{\partial u}{\partial x}(x,y) \approx \frac{u(x+h,y)-u(x-h,y)}{2h}$
  $\frac{\partial^2 u}{\partial x^2}(x,y) \approx \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}$
  $\frac{\partial u}{\partial y}(x,y) \approx \frac{u(x,y+h)-u(x,y-h)}{2h}$
  $\frac{\partial^2 u}{\partial y^2}(x,y) \approx \frac{u(x,y+h)-2u(x,y)+u(x,y-h)}{h^2}$