

## Problem 1 (TMA4125/TMA4130)

Use the Laplace transformation for solving the differential equation

$$y'' + 3y' + 2y = tu(t - 1)$$

with the initial conditions

$$y(0) = 1, \quad y'(0) = -1.$$

### Possible Solution

Application of the Laplace-transform to the equation gives us

$$s^2Y - sy(0) - y'(0) + 3sY - 3y(0) + 2Y = \mathcal{L}(tu(t - 1)).$$

The right hand side computes to

$$\begin{aligned} \mathcal{L}(tu(t - 1)) &= \mathcal{L}\left(\left((t - 1) + 1\right)u(t - 1)\right) = e^{-s}\mathcal{L}(t + 1) \\ &= e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) = e^{-s}\frac{s + 1}{s^2}. \end{aligned}$$

Inserting this and the initial conditions into the transformed equation, we obtain

$$s^2Y - s + 1 + 3sY - 3 + Y = e^{-s}\frac{s + 1}{s^2}.$$

Noting that

$$(s^2 + 3s + 1) = (s + 2)(s + 1)$$

we obtain that

$$(s + 2)(s + 1)Y = s + 2 + e^{-s}\frac{s + 1}{s^2}$$

and thus

$$Y = \frac{1}{s + 1} + e^{-s}\frac{1}{s^2(s + 2)}.$$

Now

$$\frac{1}{s^2(s + 2)} = \frac{1}{2} \frac{s + 2 - s}{s^2(s + 2)} = \frac{1}{2} \frac{1}{s^2} - \frac{1}{2} \frac{1}{s(s + 2)} = \frac{1}{2} \frac{1}{s^2} - \frac{1}{4} \frac{s + 2 - s}{s(s + 2)} = \frac{1}{2} \frac{1}{s^2} - \frac{1}{4} \frac{1}{s} + \frac{1}{4} \frac{1}{s + 2}.$$

Thus

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left(\frac{1}{s + 1} + e^{-s}\left(\frac{1}{2} \frac{1}{s^2} - \frac{1}{4} \frac{1}{s} + \frac{1}{4} \frac{1}{s + 2}\right)\right) \\ &= e^{-t} + \frac{1}{4}u(t - 1)\left(2(t - 1) - 1 + e^{-(t-2)}\right) \\ &= e^{-t} + \frac{1}{4}u(t - 1)\left(2t - 3 + e^{2-t}\right). \end{aligned}$$

## Problem 1 (TMA4122)

Let  $f$  be the function

$$f(x) = \frac{1}{1+x^2} \quad \text{for } x \in \mathbb{R}.$$

Use the Fourier transformation for computing the convolution  $(f * f)(x)$ .

### Possible Solution

We have

$$\mathcal{F}(f * f)(\omega) = \sqrt{2\pi}(\mathcal{F}(f)(\omega))^2 = \sqrt{2\pi} \left( \sqrt{\frac{\pi}{2}} e^{-|\omega|} \right)^2 = \sqrt{\frac{\pi^3}{2}} e^{-2|\omega|}.$$

Now we note that

$$\mathcal{F}^{-1}(e^{-2|\omega|})(x) = \sqrt{\frac{2}{\pi}} \frac{2}{4+x^2},$$

and therefore

$$f * f(x) = \sqrt{\frac{\pi^3}{2}} \mathcal{F}^{-1}(e^{-2|\omega|})(x) = \frac{2\pi}{4+x^2}.$$

## Problem 1 (TMA4123)

Consider the Matlab script

```
function x = TMA4123(N)
x = 0;
for i = 1:N
x = x - (exp(x) - x^2) / (exp(x) - 2*x);
end
```

Compute the return value of the script for  $N = 2$  and explain why this is an approximation to the solution of the equation

$$e^x = x^2.$$

### Possible Solution

Newton's method for the solution of the equation  $e^x - x^2 = 0$  reads

$$x_{n+1} = x_n - \frac{e^{x_n} - x_n^2}{e^{x_n} - 2x_n},$$

which is exactly the formula that defines the updates in the for-loop. Thus the Matlab script is nothing else than an implementation of Newton's method for the solution of the given equation with starting value  $x = 0$  and  $N$  iterations.

With  $N = 2$  we obtain in the first step a value of

$$x = 0 - \frac{e^0 - 0^2}{e^0 - 2 \cdot 0} = -1,$$

and in the second step

$$x = -1 - \frac{e^{-1} - (-1)^2}{e^{-1} - 2 \cdot (-1)} \approx -0.7330436.$$

This is also the return value of the script.

## Problem 2

Find the polynomial of lowest degree that interpolates the points

$$\begin{array}{c|c|c|c|c|c} x_i & -2 & -1 & 0 & 1 & 2 \\ \hline f(x_i) & 2 & 4 & 0 & -4 & 4 \end{array}$$

### Possible Solution

Using Newton interpolation<sup>1</sup>, we obtain

$$\begin{array}{c|c|c|c|c} -2 & 2 & & & \\ & 2 & & & \\ -1 & 4 & -3 & & \\ & -4 & 1 & & \\ 0 & 0 & 0 & 1/4 & \\ & -4 & 2 & & \\ 1 & -4 & 6 & & \\ & 8 & & & \\ 2 & 4 & & & \end{array}$$

Thus the interpolation polynomial in Newton form reads

$$\begin{aligned} p(x) &= 2 + 2(x + 2) - 3(x + 2)(x + 1) + (x + 2)(x + 1)x + \frac{1}{4}(x + 2)(x + 1)x(x - 1) \\ &= 2 + 2(x + 2) - 3(x^2 + 3x + 2) + x^3 + 3x^2 + 2x + \frac{1}{4}(x^4 + 2x^3 - x^2 - 2x) \\ &= \frac{1}{4}x^4 + \frac{3}{2}x^3 - \frac{1}{4}x^2 - \frac{11}{2}x. \end{aligned}$$

<sup>1</sup>Lagrange interpolation would be fine as well

### Problem 3

Use the trapezoidal rule with step length  $h = 0.25$  in order to find an approximation  $T$  of the integral

$$I = \int_0^1 e^{x^2} dx.$$

Find an upper bound for the error  $|I - T|$ .

### Possible Solution

The trapezoidal rule with  $h = 0.25$  for this integral reads

$$T = \frac{1}{4} \left( \frac{1}{2} e^0 + e^{0.25^2} + e^{0.5^2} + e^{0.75^2} + \frac{1}{2} e^1 \right) \approx 1.490679.$$

For the error estimate we use the formula

$$|\epsilon| \leq \frac{1}{12} 0.25^2 \max_{0 \leq x \leq 1} |(e^{x^2})''|.$$

Now

$$(e^{x^2})' = 2xe^{x^2}$$

and thus

$$(e^{x^2})'' = (2xe^{x^2})' = (4x^2 + 2)e^{x^2}.$$

Since both the functions  $4x^2 + 2$  and  $e^{x^2}$  are positive and increasing on the interval  $[0, 1]$ , the maximum is attained for the largest value of  $x$ , that is, for  $x = 1$ . Thus

$$\max_{0 \leq x \leq 1} |(e^{x^2})''| = 6e$$

and we obtain the error estimate

$$|\epsilon| \leq \frac{1}{12} 0.25^2 6e = \frac{e}{32} \approx 0.08494631.$$

### Problem 4

Perform two iterations of the Jacobi method for solving the linear system

$$\begin{aligned} 5x_1 + 2x_2 + x_3 &= 5, \\ -x_1 - 5x_2 + x_3 &= 5, \\ x_1 + x_2 + 3x_3 &= -3. \end{aligned}$$

Use the initial value  $x^{(0)} = (0, 0, 0)$ .

## Possible Solution

The Jacobi method for this system reads

$$\begin{aligned}x_1^{(k+1)} &= \frac{1}{5}(5 - 2x_2^k - x_3^k), \\x_2^{(k+1)} &= -\frac{1}{5}(5 + x_1^k - x_3^k), \\x_3^{(k+1)} &= \frac{1}{3}(-3 - x_1^k - x_2^k).\end{aligned}$$

In the first iteration we obtain

$$\begin{aligned}x_1^{(1)} &= \frac{1}{5}(5 - 2 \cdot 0 - 0) = 1, \\x_2^{(1)} &= -\frac{1}{5}(5 + 0 - 0) = -1, \\x_3^{(1)} &= \frac{1}{3}(-3 - 0 - 0) = -1.\end{aligned}$$

In the second iteration, we obtain

$$\begin{aligned}x_1^{(2)} &= \frac{1}{5}(5 - 2 \cdot (-1) - (-1)) = \frac{8}{5}, \\x_2^{(2)} &= -\frac{1}{5}(5 + 1 - (-1)) = -\frac{7}{5}, \\x_3^{(2)} &= \frac{1}{3}(-3 - 1 - (-1)) = -1.\end{aligned}$$

## Problem 5

Let  $f$  be the 6-periodic function defined by

$$f(x) = x + 3 \quad \text{for } -3 < x < 3.$$

Find the Fourier series of  $f$ .

## Possible Solution

The Fourier coefficient  $a_0$  computes as

$$a_0 = \frac{1}{6} \int_{-3}^3 x + 3 \, dx = 3.$$

If  $n \geq 1$ , we obtain

$$a_n = \frac{1}{3} \int_{-3}^3 (x + 3) \cos \frac{n\pi x}{3} \, dx.$$

Now note that the function  $x$  is odd, while  $\cos$  is even, which implies that  $x \cos \frac{n\pi x}{3}$  is odd, and thus its integral from  $-3$  to  $3$  equal to 0. Thus

$$a_n = \frac{1}{3} \int_{-3}^3 3 \cos \frac{n\pi x}{3} dx = \frac{3}{n\pi} \sin \frac{n\pi x}{3} \Big|_{-3}^3 = 0.$$

For the computation of the coefficients  $b_n$  we obtain

$$b_n = \frac{1}{3} \int_{-3}^3 (x + 3) \sin \frac{n\pi x}{3} dx,$$

which reduces, because  $\sin$  is odd, to

$$b_n = \frac{1}{3} \int_{-3}^3 x \sin \frac{n\pi x}{3} dx,$$

Integration by parts yields

$$\begin{aligned} b_n &= -\frac{1}{3} x \frac{3}{n\pi} \cos \frac{n\pi x}{3} \Big|_{-3}^3 - \frac{1}{3} \int_{-3}^3 \left( -\frac{3}{n\pi} \cos \frac{n\pi x}{3} \right) dx \\ &= -\frac{1}{n\pi} \left( 3(-1)^n - (-3)(-1)^n \right) + \frac{1}{n\pi} \frac{3}{n\pi} \sin \frac{n\pi x}{3} \Big|_{-3}^3 \\ &= \frac{6(-1)^{n+1}}{n\pi} \end{aligned}$$

Thus we obtain the Fourier series expansion

$$f(x) = 3 + \sum_{n=1}^{\infty} \frac{6(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{3}.$$

## Problem 6

Let  $f$  be the  $2\pi$ -periodic function given by

$$f(x) = \begin{cases} e^x & \text{for } 0 < x < \pi, \\ 0 & \text{for } -\pi < x < 0. \end{cases}$$

Assume that  $a_n$  and  $b_n$  are the Fourier coefficients of  $f$ , and denote by  $g$  and  $h$  the functions

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{and} \quad h(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

Sketch the graphs of the functions  $f$ ,  $g$ , and  $h$  on the interval  $[-2\pi, 2\pi]$ , and find the values of  $f(x)$  and  $g(x)$  in the points  $x = -\pi/2$ ,  $x = 0$ , and  $x = \pi/2$ .

## Possible Solution (without sketches)

Since the function  $g$  contains only a constant and cosine terms, it is even, that is  $g(x) = g(-x)$ . Also, the function  $h$  only contains sine terms and therefore is odd, that is,  $h(-x) = -h(x)$ . Finally, we note that

$$g(x) + h(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) = f(x)$$

(at all points  $x$  where  $f$  is continuous), as  $a_n$  and  $b_n$  are the Fourier coefficients of  $f$ . In other words,

$$\begin{aligned} g(x) + h(x) &= e^x && \text{for all } 0 < x < \pi, \\ g(x) + h(x) &= 0 && \text{for all } -\pi < x < 0. \end{aligned}$$

Now the second equation can also be written as

$$g(-x) + h(-x) = 0 \quad \text{for all } 0 < x < \pi.$$

Using the fact that  $g(-x) = g(x)$  and  $h(-x) = -h(x)$ , we thus obtain the system of equations

$$\begin{aligned} g(x) + h(x) &= e^x && \text{for all } 0 < x < \pi, \\ g(x) - h(x) &= 0 && \text{for all } 0 < x < \pi. \end{aligned}$$

Solving this system for  $g(x)$  and  $h(x)$  yields

$$g(x) = h(x) = \frac{1}{2}e^x \quad \text{for all } 0 < x < \pi.$$

Since  $g$  is even and  $h$  is odd, this means that

$$g(x) = \begin{cases} \frac{1}{2}e^x & \text{for } 0 < x < \pi, \\ \frac{1}{2}e^{-x} & \text{for } -\pi < x < 0, \end{cases}$$

and

$$h(x) = \begin{cases} \frac{1}{2}e^x & \text{for } 0 < x < \pi, \\ -\frac{1}{2}e^{-x} & \text{for } -\pi < x < 0. \end{cases}$$

In particular, we obtain

$$g(-\pi/2) = g(\pi/2) = \frac{1}{2}e^{\pi/2}$$

and

$$h(-\pi/2) = -\frac{1}{2}e^{\pi/2} \quad \text{and} \quad h(\pi/2) = \frac{1}{2}e^{\pi/2}.$$

At the point 0, the functions  $g$  and  $h$  are equal to the means of their left-hand and right-hand limits, respectively. Therefore

$$g(0) = \frac{1}{2}(g(0-0) + g(0+0)) = \frac{1}{2}(e^{-0} + e^0) = 1$$

and

$$h(x) = \frac{1}{2}(h(0-0) + h(0+0)) = \frac{1}{2}(-e^{-0} + e^0) = 0.$$

## Problem 7

We want to find a numerical solution of the partial differential equation

$$\frac{\partial u}{\partial t}(x, t) = tu(x, t) + \frac{\partial^2 u}{\partial x^2}(x, t), \quad 0 \leq x \leq 1, \quad t > 0,$$

with boundary conditions

$$u(0, t) = 0, \quad u(1, t) = 1 \quad \text{for all } t > 0$$

and initial condition

$$u(x, 0) = x \quad \text{for } 0 \leq x \leq 1.$$

Formulate an explicit method for solving this partial differential equation with the given boundary and initial conditions.

Use a step length of  $h = 1/4$  in space and perform two time steps of length  $k = 1/10$ .

## Possible Solution

Denote  $x_i := ih$  and  $t_j := jk$ , and let  $u_{i,j}$  be a numerical approximation of  $u(x_i, t_j)$ .

The left hand side of the PDE can be discretised (in order to obtain an explicit method) as

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{u_{i,j+1} - u_{i,j}}{k},$$

and the right hand side as

$$t_j u(x_i, t_j) + \frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx jk u_{i,j} + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}.$$

Thus one obtains

$$\frac{u_{i,j+1} - u_{i,j}}{k} = jk u_{i,j} + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$



or, explicitly,

$$u_{i,j+1} = u_{i,j} + k \left( jk u_{i,j} + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \right).$$

With  $h = 1/4$  and  $k = 1/10$ , this becomes

$$u_{i,j+1} = u_{i,j} + \frac{j}{100} u_{i,j} + \frac{8}{5} (u_{i-1,j} - 2u_{i,j} + u_{i+1,j}).$$

Now we use the initial condition  $u(x, 0) = x$  in order to obtain that

$$\begin{aligned} u_{0,0} &= 0, \\ u_{1,0} &= \frac{1}{4}, \\ u_{2,0} &= \frac{1}{2}, \\ u_{3,0} &= \frac{3}{4}, \\ u_{4,0} &= 1. \end{aligned}$$

Thus

$$\begin{aligned} u_{1,1} &= u_{1,0} + 0 + \frac{8}{5} (u_{0,0} - 2u_{1,0} + u_{2,0}) = \frac{1}{4}, \\ u_{2,1} &= u_{2,0} + 0 + \frac{8}{5} (u_{1,0} - 2u_{2,0} + u_{3,0}) = \frac{1}{2}, \\ u_{3,1} &= u_{3,0} + 0 + \frac{8}{5} (u_{2,0} - 2u_{3,0} + u_{4,0}) = \frac{3}{4}. \end{aligned}$$

For the next iteration, we use the boundary conditions

$$u_{0,1} = 0 \quad \text{and} \quad u_{4,1} = 1,$$

and obtain

$$\begin{aligned} u_{1,2} &= u_{1,1} + \frac{1}{100} u_{1,1} + \frac{8}{5} (u_{0,1} - 2u_{1,1} + u_{2,1}) = \frac{101}{400} = 0.2525, \\ u_{2,2} &= u_{2,1} + \frac{1}{100} u_{2,1} + \frac{8}{5} (u_{1,1} - 2u_{2,1} + u_{3,1}) = \frac{101}{200} = 0.505, \\ u_{3,2} &= u_{3,1} + \frac{1}{100} u_{3,1} + \frac{8}{5} (u_{2,1} - 2u_{3,1} + u_{4,1}) = \frac{303}{400} = 0.7575. \end{aligned}$$

## Problem 8

Use the Fourier transformation for solving the partial differential equation

$$\frac{\partial u}{\partial t}(x, t) = t \frac{\partial^2 u}{\partial x^2}(x, t), \quad x \in \mathbb{R}, \quad t > 0,$$

with initial condition

$$u(x, 0) = e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$

## Possible Solution

We apply the Fourier transformation (with respect only to the  $x$ -variable) to the given PDE and obtain

$$\frac{\partial \hat{u}}{\partial t}(\omega, t) = -t\omega^2 \hat{u}(\omega, t).$$

For every  $\omega \in \mathbb{R}$ , this is an ODE with respect to  $t$ , with solution

$$\hat{u}(\omega, t) = C(\omega)e^{-\frac{\omega^2 t^2}{2}}.$$

In order to find  $C(\omega)$ , we use the initial condition, which, after a Fourier transformation, reads as

$$\hat{u}(\omega, 0) = e^{-\frac{\omega^2}{2}}.$$

Thus

$$C(\omega) = e^{-\frac{\omega^2}{2}},$$

and we obtain

$$\hat{u}(\omega, t) = e^{-\frac{\omega^2}{2}} e^{-\frac{\omega^2 t^2}{2}} = e^{-\omega^2 \frac{t^2+1}{2}}.$$

Finally we compute the inverse Fourier transform of  $\hat{u}$  and obtain

$$u(x, t) = \mathcal{F}^{-1}(\hat{u}(\omega, t)) = \frac{1}{\sqrt{t^2+1}} e^{-\frac{x^2}{2(t^2+1)}}.$$

## Problem 9a

Given the equation

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) + 5u(x, y) = 0, \quad 0 < x < \pi, \quad 0 < y < \pi/4,$$

find all solutions of the form  $u(x, y) = F(x)G(y)$  that satisfy the boundary conditions

$$u(0, y) = 0 \quad \text{and} \quad u(\pi, y) = 0, \quad 0 < y < \pi/4.$$

## Possible Solution

Inserting the function  $u(x, y) = F(x)G(y)$  into the PDE, we obtain the equation

$$F''(x)G(y) + F(x)G''(y) + 5F(x)G(y) = 0,$$

which we rewrite as

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} - 5.$$

Now the left hand side only depends on  $x$  while the right hand side only depends on  $y$ , which implies that both sides need to be constant, say equal to  $k$  for some  $k \in \mathbb{R}$ . We thus obtain that

$$\begin{aligned} F''(x) &= kF(x), \\ G''(y) &= -(5+k)G(y). \end{aligned}$$

We now consider the different possibilities for  $k$ .

- $k = 0$ :

Here the equation for  $F$  has the solution

$$F(x) = Ax + B.$$

The boundary condition  $F(0) = 0$  implies that  $B = 0$ , and the boundary condition  $F(\pi) = 0$  then implies that  $0 = A\pi$ , and thus  $A = 0$ . Thus we only obtain the trivial solution  $F(x) = 0$ .

- $k = p^2 > 0$ :

Here the equation for  $F$  has the solution

$$F(x) = A \sinh px + B \cosh px.$$

Now the boundary condition  $F(0) = 0$  implies that  $B = 0$ , and the boundary condition  $F(\pi) = 0$  then implies that  $0 = A \sinh p\pi$ , and therefore  $A = 0$ . Again, we only obtain the trivial solution.

- $k = -p^2 < 0$ :

Here we obtain for  $F$  the solutions

$$F(x) = A \sin px + B \cos px.$$

Again the boundary condition  $F(0) = 0$  implies that  $B = 0$ . The boundary condition  $F(\pi) = 0$ , however, then implies that either  $A = 0$  or  $p \in \mathbb{N}$ . Thus we obtain the non-trivial solution

$$F(x) = \sin px$$

with  $p \in \mathbb{N}$ .

Now we turn to the solution of the equation

$$G''(y) = -(5+k)G(y) = (p^2 - 5)G(y).$$

For  $p = 1$  this has the solutions

$$G_1(y) = A \sin(2y) + B \cos(2y).$$

For  $p = 2$  we have the solutions

$$G_2(y) = A \sin(y) + B \cos(y).$$

Finally, for  $p \geq 3$  we have the solutions

$$G(y) = A \sinh(\sqrt{p^2 - 5}y) + B \cosh(\sqrt{p^2 - 5}y).$$

In total, we therefore obtain the nontrivial solutions

$$u_1(x, y) = \sin(x)(A \sin(2y) + B \cos(2y)),$$

$$u_2(x, y) = \sin(2x)(A \sin(y) + B \cos(y)),$$

$$u_p(x, y) = \sin(px)(A \sinh(\sqrt{p^2 - 5}y) + B \cosh(\sqrt{p^2 - 5}y)) \quad \text{for } p = 3, 4, \dots$$

## Problem 9b

Find the solution of the problem in part **a**) that in addition satisfies the boundary conditions

$$\begin{aligned} u(x, 0) &= \sin(x), & 0 < x < \pi, \\ u(x, \pi/4) &= \sin(x), & 0 < x < \pi. \end{aligned}$$

## Possible Solution

Using the results of Problem 9a, we obtain that

$$\begin{aligned} u(x, y) &= \sin(x)(A_1 \sin(2y) + B_1 \cos(2y)) + \sin(2x)(A_2 \sin(y) + B_2 \cos(y)) \\ &\quad + \sum_{p=3}^{\infty} \sin(px)(A_p \sinh(\sqrt{p^2 - 5}y) + B_p \cosh(\sqrt{p^2 - 5}y)) \end{aligned}$$

for some coefficients  $A_p, B_p \in \mathbb{R}$ ,  $p \in \mathbb{N}$ .

The boundary condition  $u(x, 0) = \sin(x)$  now yields that

$$\sin x = u(x, 0) = B_1 \sin(x) + B_2 \sin(2x) + \sum_{p=3}^{\infty} \sin(px) B_p,$$

which immediately implies that

$$B_1 = 1 \quad \text{and} \quad B_p = 0 \text{ for } p \geq 2.$$

That is,

$$\begin{aligned} u(x, y) = \sin(x) \cos(2y) + A_1 \sin(x) \sin(2y) + A_2 \sin(2x) \sin(y) \\ + \sum_{p=3}^{\infty} A_p \sin(px) \sinh(\sqrt{p^2 - 5}y). \end{aligned}$$

Now we insert the second boundary condition  $u(x, \pi/4) = \sin(x)$  and obtain that

$$\begin{aligned} \sin(x) = \sin(x) \cos(\pi/2) + A_1 \sin(x) \sin(\pi/2) + A_2 \sin(2x) \sin(\pi/4) \\ + \sum_{p=3}^{\infty} A_p \sin(px) \sinh(\sqrt{p^2 - 5}\pi/4). \end{aligned}$$

Since  $\cos(\pi/2) = 0$ ,  $\sin(\pi/4) = 1/\sqrt{2}$ ,  $\sin(\pi/2) = 1$ , and  $\sinh(\sqrt{p^2 - 5}\pi/4) \neq 0$  for all  $p$ , this implies that that

$$A_1 = 1 \quad \text{and} \quad A_p = 0 \text{ for } p \geq 2.$$

Thus

$$u(x, y) = \sin(x) \cos(2y) + \sin(x) \sin(2y).$$