Problem 1

The function $f(x) = 1 - x$, defined on the interval $[0, 2]$, is to be extended to an odd function g with period 4. Sketch the graph of the function g on the interval $[-4, 4]$ and find the Fourier series of g.

Possible Solution

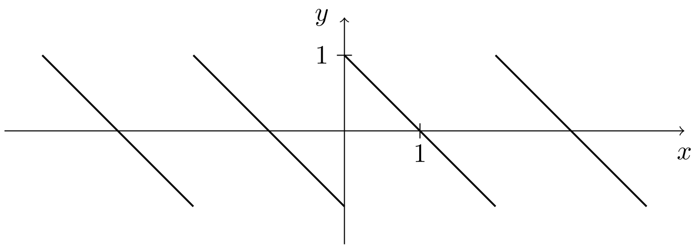
First we extend the function f to an odd function on the interval $[-2, 2]$ by setting $g(x) := -f(-x)$ for $-2 < x < 0$. This means that $g(x) = -f(-x) = -(1 - (-x)) = -1 - x$ for $-2 < x < 0$. Thus we have

$$g(x) = \begin{cases} 1 - x & \text{if } 0 < x < 2, \\ -1 - x & \text{if } -2 < x < 0. \end{cases}$$

Next,we consider the 4-periodic extension of this function. In particular, we have for $2 < x < 4$ that $g(x) = g(x - 4) = -1 - (x - 4) = 3 - x$, and for $-4 < x < -2$ we have $g(x) = g(x + 4) = 1 - (x + 4) = -3 - x$. Thus the function g is defined on the interval where we need to sketch it as

$$g(x) = \begin{cases} -3 - x & \text{if } -4 < x < -2, \\ -1 - x & \text{if } -2 < x < 0, \\ 1 - x & \text{if } 0 < x < 2, \\ 3 - x & \text{if } 2 < x < 4. \end{cases}$$

The sketch we obtain is:



One might note here that the function g is not only 4-periodic, but also 2-periodic. (Rather: Its fundamental period is 2).

Because the function g is odd, the Fourier coefficients a_n, $n = 0, 1, \ldots$, are all zero. Moreover, we obtain for the Fourier coefficients b_n the expression

$$\begin{aligned}
b_n &= \frac{2}{2}\int_0^2 (1-x)\sin\frac{n\pi x}{2}\,dx \\
&= \int_0^2 \sin\frac{n\pi x}{2}\,dx - \int_0^2 x\sin\frac{n\pi x}{2}\,dx \\
&= -\frac{2}{n\pi}(\cos n\pi - 1) - \left[\frac{n^2\pi^2}{4}\sin\frac{n\pi x}{2} - \frac{2x}{n\pi}\cos\frac{n\pi x}{2}\right]_{x=0}^{x=2} \\
&= \frac{2}{n\pi}(1-\cos n\pi) + \frac{4}{n\pi}\cos n\pi \\
&= \frac{2}{n\pi}(1-\cos n\pi + 2\cos n\pi) \\
&= \frac{2}{n\pi}(1+(-1)^n) \\
&= \begin{cases} 0 & \text{when } n \text{ is odd} \\ \frac{4}{n\pi} & \text{when } n \text{ is even.} \end{cases}
\end{aligned}$$

Thus the Fourier series we obtain is

$$\sum_{n=1}^{\infty} \frac{4}{2n\pi}\sin\frac{2n\pi x}{2},$$

or simplified slightly,

$$\sum_{n=1}^{\infty} \frac{1}{n\pi}\sin n\pi x.$$

(Note that it is also possible (and perfectly fine) to regard the function g as a 2-periodic function and compute the Fourier coefficients on the interval $[-1,1]$ instead. This changes the integration interval to $[0,1]$ and the factor in front of the integral to 2. The final series remains unchanged.)

Problem 2

The 2π-periodic function f defined as $f(x) = x^2$ for $-\pi < x < \pi$ has the Fourier coefficients

$$\begin{aligned}
a_0 &= \frac{\pi^2}{3}, \\
a_n &= (-1)^n\frac{4}{n^2}, \qquad & n &= 1,2,\dots, \\
b_n &= 0, & n &= 1,2,\dots.
\end{aligned}$$

Use this information in order to compute the sum of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Hint: Parseval's identity!

Possible Solution

We use Parseval's identity, which states that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

In this case, this means that

$$2\frac{\pi^4}{9} + \sum_{n=1}^{\infty}\frac{16}{n^4} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{2\pi^4}{5}.$$

Thus

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{16}\left(\frac{2\pi^4}{5} - \frac{2\pi^4}{9}\right) = \frac{\pi^4}{90}.$$

Problem 3

Use the Laplace transform for solving the integro-differential equation

$$y'' + y' + y = t^2 - \int_0^t y(\tau)e^{t-\tau}\,d\tau$$

with initial values

$$y(0) = 1, \qquad y'(0) = 0.$$

Possible Solution

We note that the integral on the right hand side of the equation is actually the convolution of the functions y and e^t. Application of the Laplace transform to the equation thus yields

$$s^2Y - sy(0) - y'(0) + sY - y(0) + Y = \frac{2}{s^3} - Y\cdot\mathcal{L}(e^t).$$

Using that $y(0) = 1$, $y'(0) = 0$, and $\mathcal{L}(e^t) = \frac{1}{s-1}$, we obtain the equation

$$s^2Y - s + sY - 1 + Y = \frac{2}{s^3} - Y\frac{1}{s-1}.$$

Thus

$$\left(s^2 + s + 1 + \frac{1}{s-1}\right)Y = s + 1 + \frac{2}{s^3}.$$

This can be simplified to

$$\frac{s^3}{s-1}Y = s + 1 + \frac{2}{s^3}.$$

From this we obtain that

$$Y = \frac{(s-1)(s+1)}{s^3} + \frac{2(s-1)}{s^6}$$

or

$$Y = \frac{1}{s} - \frac{1}{s^3} + 2\frac{1}{s^5} - 2\frac{1}{s^6}.$$

Thus

$$y(t) = 1 - \frac{t^2}{2} + \frac{t^4}{12} - \frac{t^5}{60}.$$

Problem 4

Perform three steps of fixed point iteration with starting value $x_0 = 0$ for solving the equation

$$x = 1 + \frac{1}{2} \arctan x.$$

Show that the iteration converges.

Possible solution

Let $g(x) = 1 + \frac{1}{2} \arctan x$. Then fixed point iteration is defined as

$$x_{n+1} = g(x_n).$$

With $x_0 = 0$ we obtain

$$\begin{aligned} x_1 &= 1, \\ x_2 &\approx 1.3927, \\ x_3 &\approx 1.4740. \end{aligned}$$

We have

$$g'(x) = \frac{1}{2(1 + x^2)}$$

and thus

$$0 < g'(x) \le \frac{1}{2} < 1$$

for all $x \in \mathbb{R}$. Thus fixed point iteration converges for all starting values.

Remark: It is *not* enough to show only that $|g'(x)| < 1$ for all x and by far not enough to show that $|g'(x_0)| < 1$. The condition that there is a constant $K < 1$ (here: $K = 1/2$) such that $|g'(x)| \le K$ for all x is crucial.

However, the condition $|g'(x)| < 1$ for all x is enough to guarantee convergence, if one has already shown by other means (e.g. the mean value theorem) that the function g actually has a fixed point. Thus, an alternative solution to the problem would be:

- The function $g(x)$ satisfies $1 - \pi/4 < g(x) < 1 + \pi/4$ for all x.
- Since g is continuous, there exists some $1 - \pi/4 < x < 1 + \pi/4$ such that $g(x) = x$ (mean value theorem).
- $|g'(x)| = 1/(2(1 + x^2)) < 1$ for all x, and thus the fixed point iteration converges.

Problem 5

Use the trapezoid method with step length $h = 0.4$ for the computation of an approximation T of the integral

$$I = \int_{-1}^{1} e^{-x} \, dx.$$

Find an upper bound for the error $|I - T|$.

Possible solution

In this case, the trapezoid rule yields

$$T = 0.4 \cdot \left[\frac{1}{2}e^{1} + e^{0.6} + e^{0.2} + e^{-0.2} + e^{-0.6} + \frac{1}{2}e^{-1}\right] \approx 2.3816.$$

The error can be estimated by

$$|T - I| \leq \frac{2}{12} 0.4^2 \max_{-1 \leq x \leq 1} |f''(x)|$$

with $f(x) = e^{-x}$ and thus $f''(x) = e^{-x}$, implying that

$$\max_{-1 \leq x \leq 1} |f''(x)| = e.$$

Thus

$$|T - I| \leq \frac{1}{6} 0.4^2 e \approx 0.07249.$$

Problem 6a

Given the equation

$$\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi,\ t > 0,$$

find all solutions of the form $u(x,t) = F(x)G(t)$ that satisfy the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}(\pi,t) = 0, \qquad\qquad t > 0.$$

Possible solution

Inserting $F(x)G(t)$ into the PDE, we obtain

$$F\ddot{G} - 2F\dot{G} + FG = F''G,$$

which we can write as

$$F(\ddot{G} - 2\dot{G} + G) = F''G.$$

Dividing by F and G this becomes

$$\frac{\ddot{G} - 2\dot{G} + G}{G} = \frac{F''}{F},$$

which necessarily has to be a constant, say $k \in \mathbb{R}$. Thus we obtain the two ODEs

$$F'' = kF,$$
$$\ddot{G} - 2\dot{G} + G = kG.$$

Moreover, the boundary conditions imply that

$$F'(0) = 0 \qquad \text{and} \qquad F'(\pi) = 0.$$

We now have three possibilities:

- $k = p^2 > 0$ with $p > 0$: Here

  $$F(x) = Ae^{px} + Be^{-px}.$$

  The condition $F'(0) = 0$ gives

  $$0 = pA - pB$$

  and thus

  $$A = B.$$

  Now the condition $F'(\pi) = 0$ implies that

  $$pAe^{p\pi} - pAe^{-p\pi} = 0$$

  or

  $$pA(e^{p\pi} - e^{-p\pi}) = 0.$$

  Since $e^{p\pi} > 1$ and $e^{-p\pi} < 1$, this implies that

  $$A = 0.$$

  Thus we only have trivial solutions.

- $k = 0$: Here

  $$F(x) = A + Bx.$$

  Now the condition $F' = 0$ implies that

  $$B = 0.$$

  However, the function $F(x) = A$ satisfies both boundary conditions, and thus we have found a non-trivial solution. Now we solve the equation for G, which, for $k = 0$ gives

  $$\ddot{G} - 2\dot{G} + G = 0.$$

  The solutions of the equation are of the form

  $$G(t) = Ce^t + Dte^t.$$

  Thus we obtain the solutions

  $$F(x)G(t) = Ce^t + Dte^t.$$

- $k = -p^2 < 0$: Here

  $$F(x) = A\cos(px) + B\sin(px).$$

  The condition $F'(0) = 0$ implies that

  $$0 = F'(0) = -pA\sin(0) + Bp\cos(0) = Bp.$$

  Thus $B = 0$ and

  $$F(x) = A\cos(px).$$

Now the condition $F'(\pi) = 0$ implies that

$$-Ap\sin(p\pi) = 0,$$

and thus we obtain the non-trivial solutions

$$F(x) = A\cos(px) \qquad \text{for } p = 1, 2, \ldots.$$

Now we compute the solution of the ODE for G, which, with $k = -p^2$ reads as

$$\ddot{G} - 2\dot{G} + G = -p^2 G.$$

This equation has the solutions

$$G(t) = e^t\big(C\cos(pt) + D\sin(pt)\big).$$

Thus we obtain the solutions (ignoring the unnecessary constant A)

$$F(x)G(t) = e^t\big(C\cos(pt) + D\sin(pt)\big)\cos(px).$$

All in all, we have found the solutions

$$F(x)G(t) = Ce^t + Dte^t$$

and

$$F(x)G(t) = e^t\big(C\cos(pt) + D\sin(pt)\big)\cos(px) \qquad \text{with } p = 1, 2, \ldots.$$

Remark: It is perfectly fine and viable to solve the ODE for G using the Laplace transform: Denoting $Y = \mathcal{L}(G)$, the equation

$$\ddot{G} - 2\dot{G} + (1 + p^2)G = 0$$

with $p > 0$ transforms to

$$s^2 Y - sG(0) - \dot{G}(0) - 2sY + 2G(0) + (1 + p^2)Y = 0,$$

which can be simplified as

$$(s^2 - 2s + (1 + p^2))Y = sG(0) - 2G(0) + \dot{G}(0)$$

or

$$((s-1)^2 + p^2)Y = sG(0) - 2G(0) + \dot{G}(0).$$

Thus

$$\begin{aligned} Y &= \frac{s}{(s-1)^2 + p^2}G(0) + \frac{1}{(s-1)^2 + p^2}(\dot{G}(0) - 2G(0)) \\ &= \frac{s-1}{(s-1)^2 + p^2}G(0) + \frac{1}{(s-1)^2 + p^2}(\dot{G}(0) - G(0)). \end{aligned}$$

Applying the inverse Laplace transform, we obtain that

$$G(t) = e^t\cos(pt)G(0) + e^t\sin(pt)\frac{\dot{G}(0) - G(0)}{p}.$$

Problem 6b

Find the solution of the problem in item **a)** that additionally satisfies the initial conditions

$$u(x,0) = x^2, \quad 0 < x < \pi,$$
$$\frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < \pi.$$

(You can use the Fourier series from problem 2.)

Possible solution

The general solution of the equation has the form

$$u(x,t) = C_0 e^t + D_0 t e^t + \sum_{p=1}^{\infty} e^t \big(C_p \cos(pt) + D_p \sin(pt)\big) \cos(px).$$

The initial condition $u(x,0) = x^2$ thus implies that

$$x^2 = C_0 + \sum_{p=1}^{\infty} C_p \cos(px) \qquad \text{for } 0 < x < \pi.$$

Thus the coefficients C_0 and C_p are the Fourier-cosine coefficients of the function x^2. From problem 2 we thus obtain that

$$C_0 = \frac{\pi^2}{3},$$
$$C_p = (-1)^p \frac{4}{p^2}, \qquad p = 1, 2, \dots.$$

In order to use the second initial condition, we compute

$$\frac{\partial u}{\partial t}(x,t) = C_0 e^t + D_0(te^t + e^t) + \sum_{p=1}^{\infty} e^t \big(C_p \cos(pt) + D_p \sin(pt)\big) \cos(px)$$
$$+ \sum_{p=1}^{\infty} e^t \big(-pC_p \sin(pt) + pD_p \cos(pt)\big) \cos(px).$$

Now the initial condition $\partial_t u(x,0) = 0$ implies that

$$0 = C_0 + D_0 + \sum_{p=1}^{\infty} (C_p + pD_p) \cos(px).$$

Thus

$$D_0 = -C_0$$

and

$$D_p = -\frac{C_p}{p}, \qquad p = 1, 2, \dots.$$

Thus we see that

$$u(x,t) = \frac{\pi^2}{3} e^t - \frac{\pi^2}{3} t e^t + 4e^t \sum_{p=1}^{\infty} (-1)^p \cos(px) \Big(\frac{1}{p^2} \cos(pt) - \frac{1}{p^3} \sin(pt)\Big).$$

Problem 7

We want to find a numerical solution of the partial differential equation

$$2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2y - x, \qquad 0 < x < 1,\ 0 < y < 1,$$

with boundary conditions

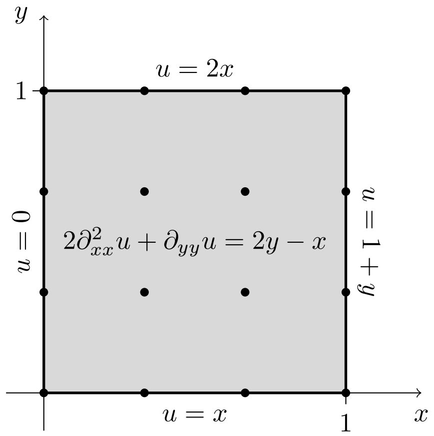
$$\begin{aligned} &u(0,y) = 0, \quad u(1,y) = 1 + y, && 0 \le y \le 1, \\ &u(x,0) = x, \quad u(1,x) = 2x, && 0 \le x \le 1. \end{aligned}$$

Set up a linear system for finding an approximation of the solution u in the points $(x_i, y_i) = (i \cdot h, j \cdot h)$ with $h = 1/3$.

(It is not necessary to solve the system.)

Possible solution

The problem we are looking at



The standard discretization of the second derivatives with finite differences gives us the equations

$$\frac{2}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + \frac{1}{h^2}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}) = 2y_j - x_i,$$

or, inserting $h = 1/3$, $y_j = j/3$, $x_i = i/3$,

$$18u_{i-1,j} - 54u_{i,j} + 18u_{i+1,j} + 9u_{i,j-1} + 9u_{i,j+1} = \frac{2j}{3} - \frac{i}{3}.$$

Moreover we have the boundary conditions

$$u_{1,3} = \frac{2}{3}, \quad u_{2,3} = \frac{4}{3},$$

$$u_{0,2} = 0, \qquad u_{3,2} = \frac{5}{3},$$

$$u_{0,1} = 0, \qquad u_{3,1} = \frac{4}{3},$$

$$u_{1,0} = \frac{1}{3}, \quad u_{2,0} = \frac{2}{3}.$$

Thus we obtain the following equations:

- At u_{11}:
$$-54u_{11} + 18u_{21} + 3 + 9u_{12} = \frac{1}{3}.$$
- At u_{21}:
$$18u_{11} - 54u_{21} + 24 + 6 + 9u_{22} = 0.$$
- At u_{12}:
$$-54u_{12} + 18u_{22} + 9u_{11} + 6 = 1.$$
- At u_{22}:
$$18u_{12} - 54u_{22} + 30 + 9u_{21} + 12 = \frac{2}{3}.$$

Or, if we want to write the system in matrix form

$$\begin{pmatrix} -54 & 18 & 9 & 0 \\ 18 & -54 & 0 & 9 \\ 9 & 0 & -54 & 18 \\ 0 & 9 & 18 & -54 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{pmatrix} = \begin{pmatrix} -\frac{8}{3} \\ -30 \\ -5 \\ -\frac{124}{3} \end{pmatrix}.$$

Problem 8

Let $y(x)$ be the function that solves the ordinary differential equation

$$y'' = \cos(\pi x/2) - y^2$$

with initial conditions

$$y(0) = 1, \qquad y'(0) = 0.$$

Rewrite this second order equation as a system of first order differential equations, and use the improved Euler method with step length $h = 1$ for approximating the value of $y(x)$ in the point $x = 2$.

Possible Solution

We rewrite the equation as

$$\begin{aligned} z_1' &= z_2, & z_1(0) &= 1, \\ z_2' &= \cos(\pi x/2) - z_1^2, & z_2(0) &= 0. \end{aligned}$$

Or, $z' = F(x, z)$ with

$$F(x, z) = \begin{pmatrix} z_2 \\ \cos(\pi x/2) - z_1^2 \end{pmatrix}.$$

Now the iterations for the improved Euler method with $h = 1$ read as:

- First step:
$$k_1 = F(0,(1,0)) = \begin{pmatrix} 0 \\ \cos(0) - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
and
$$k_2 = F(1,(1,0)+k_1) = F(1,(1,0)) = \begin{pmatrix} 0 \\ \cos(\pi/2) - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$
Thus
$$z^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2}(k_1 + k_2) = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}.$$

- Second step:
$$k_1 = F(1,(1,-1/2)) = \begin{pmatrix} -1/2 \\ \cos(\pi/2) - 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1 \end{pmatrix}$$
and
$$k_2 = F(2,(1,-1/2)+(-1/2,-1)) = F(2,(1/2,-3/2)) = \begin{pmatrix} -3/2 \\ \cos(\pi) - (1/2)^2 \end{pmatrix} = \begin{pmatrix} -3/2 \\ -5/4 \end{pmatrix}.$$
Thus
$$z^{(2)} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1/2 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -3/2 \\ -5/4 \end{pmatrix} = \begin{pmatrix} 0 \\ -13/8 \end{pmatrix}.$$
The value we are interested in is $z_2^{(2)} = 0$. Thus $y(2) \approx 0$.

Problem 9

Perform two iterations of the Gauss–Seidel method for solving the linear system
$$\begin{aligned} 4x_1 - x_2 - x_3 &= 4, \\ x_1 + 2x_2 &= -1, \\ -x_1 - x_2 + 3x_3 &= -3. \end{aligned}$$
Use the starting value $x^{(0)} = (0,0,0)$.

Possible Solution

The Gauß–Seidel method for this problem reads as
$$\begin{aligned} x_1^{(k+1)} &= 1 + \frac{1}{4}x_2^{(k)} + \frac{1}{4}x_3^{(k)}, \\ x_2^{(k+1)} &= -\frac{1}{2} - \frac{1}{2}x_1^{(k+1)}, \\ x_3^{(k+1)} &= -1 + \frac{1}{3}x_1^{(k+1)} + \frac{1}{3}x_2^{(k+1)}. \end{aligned}$$

Thus, with $x^{(0)} = (0,0,0)$, we obtain

$$
\begin{aligned}
x_1^{(1)} &= 1, \\
x_2^{(1)} &= -\frac{1}{2} - \frac{1}{2} = -1, \\
x_3^{(1)} &= -1 + \frac{1}{3} - \frac{1}{3} = -1,
\end{aligned}
$$

and

$$
\begin{aligned}
x_1^{(2)} &= 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}, \\
x_2^{(2)} &= -\frac{1}{2} - \frac{1}{2}\frac{1}{2} = -\frac{3}{4}, \\
x_3^{(2)} &= -1 + \frac{1}{3}\frac{1}{2} - \frac{1}{3}\frac{3}{4} = -\frac{13}{12}.
\end{aligned}
$$