ONE-DIMENSIONAL HEAT EQUATION

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In the following, we will try to describe the changes of the temperature occuring in a thin metal wire. To that end, we assume for simplicity that no heat energy is lost through the surface of the wire, which is a reasonable assumption if the wire is surrounded by air (which is a very bad heat conductor) and the temperature of the wire is not too large (if the wire gets very hot, the energy loss due to radiation becomes significant).

The starting point for the derivation of an equation describing the temperature is *Fourier's law*, which states that the amount of heat energy exchanged between two different objects in a fixed amount of time is proportional to their temperature difference (and the energy always flows from the hotter object to the cooler one). Choosing the "different objects" to be two points on the wire that are infinitesimally close to each other, this means that the heat flux is proportional to the derivative of the temperature.

Denote now by $T(x,t)$ the temperature at a point x on the wire (we can model the thin wire as a one-dimensional object) at time t, and by $J(x,t)$ the density of the heat flux at the same position and time, that is, the heat flux per unit area and unit time. Then Fourier's law states that J is proportional to the space derivative of T, with a proportionality constant $k(x)$—the heat conductivity of the wire—that may depend on x. Thus we obtain the equation

$$J(x,t) = -k(x)\frac{\partial T}{\partial x}(x,t).$$

Here k is always positiv and the negative sign in the equation indicates that heat flows from hot regions to cooler regions.

As a next step, we assume that the heat energy $Q(x,t)$ that is stored in a specific point x in the wire is proportional to the temperature at the same point, the proportionality constant depending on some material constant $c(x)$ (the specific heat capacity) and the density $\rho(x)$ of the material. That is,

$$Q(x,t) = c(x)\rho(x)T(x,t). \tag{1}$$

Now consider the piece of wire between the two points x and $x+\Delta x$ (for some small $\Delta x > 0$). The total heat energy stored in this piece of wire is simply

$$\int_x^{x+\Delta x} Q(s,t)\,ds.$$

Moreover, the total heat flow in direction of the wire at the point x is given by

$$\frac{\partial Q}{\partial t}(x,t) = A\Big(-k(x)\,\frac{\partial T}{\partial x}(x,t)\Big)$$

with A being the cross-sectional area of the wire, and the heat flow at the point $x+\Delta x$ is given by

$$\frac{\partial Q}{\partial t}(x+\Delta x,t) = A\Big(-k(x+\Delta x)\,\frac{\partial T}{\partial x}(x+\Delta x,t)\Big).$$

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Since we assume that no energy is exchanged between the wire and the surrounding air and no energy is produced in the wire, it follows that the total change of energy in the section of the wire between the points x and $x + \Delta$ is given by

$$\frac{\partial}{\partial t}\int_x^{x+\Delta x} Q(s,t)\,ds = \Big(A\,k(x+\Delta x)\,\frac{\partial T}{\partial x}(x+\Delta x,t) - A\,k(x)\,\frac{\partial T}{\partial x}(x,t)\Big).$$

Now we divide this equation by Δx and take the limit $\Delta x \to 0$. Then we obtain the equation

$$\frac{\partial Q}{\partial t}(x,t) = \frac{\partial}{\partial x}\Big(A\,k(x)\frac{\partial T}{\partial x}(x,t)\Big).$$

Combining this with (1) provides the one-dimensional heat equation

$$c(x)\rho(x)\frac{\partial T}{\partial t}(x,t) = \frac{\partial}{\partial x}\Big(A\,k(x)\frac{\partial T}{\partial x}(x,t)\Big).$$

In the special case where the wire has a constant density ρ, a constant heat conductivity k, and a constant specific heat capacity c (that is, the functions ρ, k and c are actually constants), this simplifies to

$$\frac{\partial T}{\partial t}(x,t) = \kappa\frac{\partial^2 T}{\partial x^2}(x,t)$$

with

$$\kappa = \frac{A\,k}{\rho c}.$$

This constant κ is called the thermal diffusivity of the wire.

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