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TMA4130 Mathematics 4N, fall 2014

Solution suggestions

1a First we must observe that $f(x)$ is an even function, so the Fourier series of $f(x)$ will be of the form

$$a_0 + \sum_{i=1}^{\infty} a_n \cos(nx).$$

So the computation follows to

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx = \left.\frac{-\cos(x)}{\pi}\right]_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin((n+1)x) - \sin((n-1)x)\,dx$$

First if $n = 1$, we have that

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin(2x) - \sin(0x)\,dx = \left.\frac{-\cos(2x)}{2\pi}\right]_0^{\pi} = 0$$

and for $n \geq 2$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin((n+1)x) - \sin((n-1)x)\,dx = \frac{1}{\pi}\left.\left(\frac{-\cos((n+1)x)}{n+1} + \frac{\cos((n-1)x)}{n-1}\right)\right]_0^{\pi}$$

$$= \begin{cases} \frac{-4}{\pi}\frac{1}{n^2-1} & \text{if n is even,} \\ 0 & \text{if n is odd.} \end{cases}$$

Therefore we have that the Fourier series of $f(x)$ is

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{\text{n even}} \frac{1}{n^2-1}\cos(nx) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos(2nx)$$

1b First observe that $f(x)$ is continuous at 0, so we have that

$$0 = f(0) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos(2n0) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1},$$

therefore

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$$

2 The polynomial in the denominator factorizes as

$$s^2 - 2s - 3 = (s+1)(s-3).$$

Thus

$$\frac{11-s}{s^2-2s-3} = \frac{11-s}{(s+1)(s-3)}.$$

We now make the ansatz

$$\frac{11-s}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s-3}.$$

Multiplying this equation with $(s+1)(s-3)$ we obtain

$$11 - s = (s-3)A + (s+1)B = s(A+B) - 3A + B.$$

Thus

$$A + B = -1,$$
$$-3A + B = 11.$$

From this we see that $A = -3$ and $B = 2$. That is,

$$F(s) = \frac{11-s}{(s+1)(s-3)} = -\frac{3}{s+1} + \frac{2}{s-3}.$$

Now we see that

$$f(t) = -3e^{-t} + 2e^{3t}.$$

3 Applying the Laplace transformation to the differential equation yields

$$s^2Y - sy(0) - y'(0) = Y + e^{-s} + 2\frac{e^{-s}}{s-1}.$$

Using the initial conditions we obtain

$$(s^2-1)Y = 1 + e^{-s} + 2\frac{e^{-s}}{s-1} = 1 + e^{-s}\frac{s+1}{s-1}.$$

Dividing by $s^2 - 1$, this results in

$$Y = \frac{1}{s^2-1} + e^{-s}\frac{s+1}{(s-1)(s^2-1)} = \frac{1}{s^2-1} + e^{-s}\frac{1}{(s-1)^2}.$$

Applying the inverse Laplace transform, this becomes

$$y(t) = \sinh(t) + (t-1)e^{t-1}u(t-1).$$

4a Since $u = F(x)G(t)$ we have that $u_{xx} = F''G$ and $u_t = FG'$, and inserting them in the equation we have $FG' = 2F''G$. Then since

$$\frac{G'}{G} = 2\frac{F''}{F} = k \qquad \text{where k is a constant,}$$

we can derive the two ODEs

$$\begin{cases} F'' - \frac{k}{2}F = 0 \\ G' - kG = 0 \end{cases}$$

The boundary conditions say that $F'(0) = 0$ and $F(\pi) = 0$, so we can easily check that if $k = 0$ or $k > 0$ then $F = 0$, and hence $u = 0$. Since we want only non-trivial solutions, we neglect these options.

So we consider $k = -p^2 < 0$, and hence the solutions to the ODE $F'' + \frac{p^2}{2}F = 0$ are of the form

$$F(x) = A\cos\left(\frac{p}{\sqrt{2}}x\right) + B\sin\left(\frac{p}{\sqrt{2}}x\right).$$

Since $F'(0) = -A\frac{p}{\sqrt{2}}\sin\left(\frac{p}{\sqrt{2}}0\right) + B\frac{p}{\sqrt{2}}\cos\left(\frac{p}{\sqrt{2}}0\right) = B\frac{p}{\sqrt{2}} = 0$, we have $B = 0$. Finally since $F(\pi) = A\cos\left(\frac{p}{\sqrt{2}}\pi\right) = 0$, this implies $\cos\left(\frac{p}{\sqrt{2}}\pi\right) = 0$, so $\frac{p}{\sqrt{2}}\pi = \frac{2n+1}{2}\pi$ for $n = 0, 1, \ldots$. So if we isolate p we have $p = \frac{2n+1}{\sqrt{2}}$ for $n = 0, 1, \ldots$, so we have that

$$F_n(x) = \cos\left(\frac{2n+1}{2}x\right) \qquad n = 0, 1, \ldots$$

So we solve the second ODE, so $G' + p^2G = 0$ and replacing $p = \frac{2n+1}{\sqrt{2}}$ we have that

$$G_n' + \frac{(2n+1)^2}{2}G_n = 0 \qquad n = 0, 1, \ldots$$

and easily we can find the solution

$$G_n = A_n e^{-\frac{(2n+1)^2}{2}t} \qquad n = 0, 1, \ldots$$

Therefore merging both solutions we have that

$$u_n = F_nG_n = A_n e^{-\frac{(2n+1)^2}{2}t}\cos\left(\frac{2n+1}{2}x\right) \qquad n = 0, 1, \ldots$$

4b The initial conditions say that $u(x,0) = \cos(x)\cos(\frac{3}{2}x) = \frac{1}{2}\left(\cos(\frac{1}{2}x) + \cos(\frac{5}{2}x)\right)$, so

$$u(x,0) = \sum_{n=0}^{\infty} u_n(x,0) = \sum_{n=0}^{\infty} F_n(x)G_n(0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{2n+1}{2}x\right).$$

Therefore $A_0 = A_2 = \frac{1}{2}$, and hence

$$u(x,t) = \frac{1}{2}e^{-\frac{1}{2}t}\cos\left(\frac{1}{2}x\right) + \frac{1}{2}e^{-\frac{25}{2}t}\cos\left(\frac{5}{2}x\right)$$

5a We use the Lagrange polynomial formula

$$p_4(x) = 0\cdot\frac{(x+1)x(x-1)(x-2)}{(-2+1)(-2)(-2-1)(-2-2)} + 2\cdot\frac{(x+2)x(x-1)(x-2)}{(-1+2)(-1)(-1-1)(-1-2)} + 0\cdot\frac{(x+2)(x-1)(x+1)(x-2)}{(0+2)(0-1)(0+1)(0-2)}$$
$$+1\cdot\frac{(x+2)(x+1)x(x-2)}{(1+2)(1+1)1(1-2)} + 0\cdot\frac{(x+2)(x+1)x(x-1)}{(2+2)(2+1)2(2-1)}$$
$$= -\frac{1}{3}(x+2)x(x-1)(x-2) - \frac{1}{6}(x+2)(x+1)x(x-2) = -\frac{1}{2}x^4 + \frac{1}{6}x^3 + 2x^2 - \frac{2}{3}x$$

5b Using the Error formula of the Simpson method we have the following inequality

$$|E_n| \le h^4\frac{(3-1)}{180}\max_{[1,3]}|f^{iv)}(x)| < 0.001$$

We have that

$$f(x) = x^2\ln(x) \qquad f'(x) = 2x\ln(x) + x \qquad f''(x) = 2\ln(x) + 3 \qquad f'''(x) = \frac{2}{x} \qquad \text{and} \qquad f^{(iv)}(x) = -\frac{2}{x^2}$$

so we have that $\max_{[1,3]}|-\frac{2}{x^2}| = 2$. Therefore

$$h^4\frac{2}{180}2 < 0.001$$

or

$$h^4 < 0.045$$

and thus

$$h < 0.461.$$

Since in Simpson's rule the length of the interval (in our case: $b-a=2$) has to be an even multiple of the step length h, the largest possible step size that works for Simpson's method is $h = 1/3$. With this we obtain

$$\int_1^3 x^2\ln(x)\,dx \approx \frac{1}{9}\Big[1\ln(1) + 4\Big(\frac{4}{3}\Big)^2\ln\Big(\frac{4}{3}\Big) + 2\Big(\frac{5}{3}\Big)^2\ln\Big(\frac{5}{3}\Big) + 4\Big(\frac{6}{3}\Big)^2\ln\Big(\frac{6}{3}\Big) + 2\Big(\frac{7}{3}\Big)^2\ln\Big(\frac{7}{3}\Big)$$
$$+ 4\Big(\frac{8}{3}\Big)^2\ln\Big(\frac{8}{3}\Big) + \Big(\frac{9}{3}\Big)^2\ln\Big(\frac{9}{3}\Big)\Big] \approx 6.9985.$$

6a Using the notation of the text, we have that $(x_1,y_1) = (1,1)$, $(x_2,y_1) = (2,1)$ and $(x_1,y_2) = (1,2)$, so we write $u_{11} = u(1,1)$, $u_{12} = u(1,2)$ and $u_{21} = u(2,1)$. The equation gives us the following relations

$$\frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2} + \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{h^2} = 2(ih)(jh)$$

so if we replace $h = 1$ we have

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 2ij$$

Therefore we set up the 3 equations with $(i,j) = (1,1)$ and $(i,j) = (1,2)$ and $(i,j) = (2,1)$

$$\begin{cases} u_{21}+u_{01}+u_{12}+u_{10}-4u_{11}=2 \\ u_{22}+u_{02}+u_{13}+u_{11}-4u_{12}=4 \\ u_{31}+u_{11}+u_{22}+u_{20}-4u_{21}=4 \end{cases}$$

So using the boundary conditions we have $u_{0,1}=u_{1,0}=u_{0,2}=u_{2,0}=1$, and $u_{2,2}=\frac{2^2+2^2}{16}=\frac{1}{2}$, $u_{1,3}=\frac{1+3^2}{16}=\frac{5}{8}$ and $u_{3,1}=\frac{3^2+1}{16}=\frac{5}{8}$, so the desired system is

$$\begin{cases} u_{21}+1+u_{12}+1-4u_{11}=2 \\ \frac{1}{2}+1+\frac{5}{8}+u_{11}-4u_{12}=4 \\ \frac{5}{8}+u_{11}+\frac{1}{2}+1-4u_{21}=4 \end{cases}$$

and

$$\begin{cases} u_{21}+u_{12}-4u_{11}=0 \\ u_{11}-4u_{12}=\frac{15}{8} \\ u_{11}-4u_{21}=\frac{15}{8} \end{cases}$$

6b We set up the system

$$\begin{cases} u_{21}+u_{12}-4u_{11}=0 \\ u_{11}-4u_{12}=\frac{15}{8} \\ u_{11}-4u_{21}=\frac{15}{8} \end{cases}$$

in the Gauss-Seidel form

$$\begin{cases} -\frac{1}{4}u_{21}-\frac{1}{4}u_{12}+u_{11}=0 \\ -\frac{1}{4}u_{11}+u_{12}=-\frac{15}{32} \\ -\frac{1}{4}u_{11}+u_{21}=-\frac{15}{32} \end{cases}$$

so the Gauss-Seidel iteration formula is

$$\begin{aligned} u_{11}^{(n+1)} &= \frac{1}{4}u_{21}^{(n)}+\frac{1}{4}u_{12}^{(n)} \\ u_{12}^{(n+1)} &= \frac{1}{4}u_{11}^{(n+1)}-\frac{15}{32} \\ u_{21}^{(n+1)} &= \frac{1}{4}u_{11}^{(n+1)}-\frac{15}{32} \end{aligned}$$

So if we start with $u_{11}^{(0)}=u_{12}^{(0)}=u_{21}^{(0)}=0$

$$\begin{aligned} u_{11}^{(1)} &= \frac{1}{4}0+\frac{1}{4}0=0 \\ u_{12}^{(1)} &= \frac{1}{4}0-\frac{15}{32}=-\frac{15}{32} \\ u_{21}^{(1)} &= \frac{1}{4}0-\frac{15}{32}=-\frac{15}{32} \end{aligned}$$

and continue with the second step

$$\begin{aligned} u_{11}^{(2)} &= \frac{1}{4}\left(-\frac{15}{32}\right)+\frac{1}{4}\left(-\frac{15}{32}\right)=-\frac{15}{64} \\ u_{12}^{(2)} &= \frac{1}{4}\left(-\frac{15}{64}\right)-\frac{15}{32}=-\frac{135}{256} \\ u_{21}^{(2)} &= \frac{1}{4}\left(-\frac{15}{64}\right)-\frac{15}{32}=-\frac{135}{256} \end{aligned}$$

and finally the

$$
\begin{aligned}
u_{11}^{(3)} &= \frac{1}{4}\left(-\frac{135}{256}\right) + \frac{1}{4}\left(-\frac{135}{256}\right) = -\frac{135}{512} \\
u_{12}^{(3)} &= \frac{1}{4}\left(-\frac{135}{512}\right) - \frac{15}{32} = -\frac{1095}{2048} \\
u_{21}^{(3)} &= \frac{1}{4}\left(-\frac{135}{512}\right) - \frac{15}{32} = -\frac{1095}{2048}
\end{aligned}
$$